SINGULAR HAMMERSTEIN EQUATIONS AND MAXIMAL MONOTONE OPERATORS

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Consider the nonlinear integral equation of Hammerstein type

\[ u(x) + \int_{\Omega} k(x, y)f(y, u(y))\vartheta(dy) = h(x) \quad (x \in \Omega), \]

where \( h \) and the solution \( u \) lie in a space \( X \) of measurable functions on \( \Omega \). The Hammerstein equation is said to be regular if for

\[ F(u)(y) = f(y, u(y)) \quad (y \in \Omega); \quad Kv(x) = \int_{\Omega} k(x, y)v(y)\vartheta(dy) \quad (x \in \Omega), \]

the operator \( KF \) is defined on all of \( X \), and singular otherwise.

In some recent papers (summarized in [2]), the writers have studied the existence theory for regular Hammerstein equations in \( L^p(\vartheta) \) with \( 1 < p \leq +\infty \) under very general assumptions on \( K \) and \( F \). In later papers (cf. [4]), one of the writers has obtained general existence results for the singular case, using measure-theoretic arguments and mild compactness assumptions on \( K \). We present results here without compactness assumptions based on a new theorem on linear monotone operators.

**Theorem 1.** Let \( X \) be a reflexive Banach space, \( L_0 \) and \( L_1 \) linear monotone mappings from \( X \) into \( 2^{X^*} \) with \( L_0 \subseteq L_1 \). Then there exists a maximal monotone linear map from \( X \) into \( 2^{X^*} \) such that \( L_0 \subseteq L \subseteq L_1 \).

For single-valued, densely defined maps in Hilbert space, this coincides with a theorem of R. S. Phillips [6] obtained using ideas of M. Krein [5]. For reflexive Banach spaces, in general, we have as a corollary a result obtained in 1968 by one of the writers [1]:

**Theorem 2.** Let \( X \) be a reflexive Banach space, \( L \) a closed linear monotone map from \( X \) into \( 2^{X^*} \). Then \( L \) is maximal monotone if and only if \( L^* \) is monotone.

We sketch the proof of Theorem 1 (detailed proofs are given in [3]). By a Zorn's Lemma argument we may construct a monotone linear map \( L \) with \( L_0 \subseteq L \subseteq L_1^* \) such that \( L \) is maximal monotone in the graph of \( L_1^* \). Let \( J \) be a duality map of \( X \) into \( X^* \) corresponding to a norm on \( X \) with \( X \) and \( X^* \) locally uniformly convex.

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Let $w_0$ be any element of $X^*$. It suffices to find $u_0$ in $X$ such that $w_0 \in (L + J)(u_0)$. For each finite-dimensional subspace $M$ of $X$, let $\xi_M$ be the injection map of $M$ into $X$, $\xi_M^*: X^* \to M^*$. We form linear monotone mappings $L_M$ and $L_{1,M}$ of $M$ into $2^{M^*}$ with $L_M \subseteq (L_{1,M})^*$ by

$$L_M(x) = \xi_M^*(L(x)), \quad L_{1,M}(x) = \xi_M^*(L_1(x)).$$

We apply the multivalued finite-dimensional version of Phillips' theorem (a simple direct proof for which is given in [3]) to obtain a maximal monotone mapping $K_M$ from $M$ to $2^{M^*}$ such that $L_M \subseteq K_M \subseteq (L_{1,M})^*$. Hence, we may find $u_M$ in $M$ such that $\xi_M^*(w_0) \in K_M(u_M) + \xi_M^*(J(u_M))$.

For each $[u, w]$ in $G(L)$ and for each $[x, y]$ in $G(L)$, we have

$$\langle w_0 - J(u_M), u \rangle = \langle w, u_M \rangle,$$

(3)

$$\langle y + J(u_M) - w_0, x - u_M \rangle \geq 0.$$

(4)

The elements $\{[u_M, J(u_M)]\}$ are bounded since $J$ is coercive. Since $X$ is reflexive, we may assume a filter $\{[u_M, J(u_M)]\}$ converging weakly to $[u_0, y_0]$ in $X \times X^*$. Since equality (3) holds eventually for each $[u, w]$ in $G(L_1)$, we may take the limit to find that $\langle w_0 - y_0, u \rangle = \langle w, u_0 \rangle$ for all $[u, w]$ in $G(L_1)$. Hence $[u_0, w_0 - y_0]$ lies in $G(L_1^*)$. From inequality (4) which holds eventually for each $[x, y]$ in $G(L)$, we obtain

$$\lim \langle J((u_M), u_M) \rangle - \langle y_0, u_0 \rangle \leq \langle y + y_0 - w_0, x - u_0 \rangle.$$

(5)

Since $J$ is pseudo-monotone, the left side is nonnegative. Since $[u_0, w_0 - y_0] \in G(L_1^*)$ and $L$ is assumed maximal monotone in $G(L_1^*), [u_0, w_0 - y_0]$ lies in $G(L)$. Replacing $[x, y]$ by this element, it follows that the left side of (5) is zero, and hence $y_0 = J(u_0)$. Thus $w_0 - J(u_0) \in L(u_0)$, i.e. $w_0 \in (L + J)(u_0)$. Q.E.D.

The application to singular Hammerstein equations is made through the following more general theorem:

**Theorem 3.** Let $\beta$ be a finite measure on $\Omega$, $X$ a reflexive Banach space with $L^\infty(\beta) \subseteq X \subseteq L^1(\beta)$, $L^\infty(\beta) \subseteq X^* \subseteq L^1(\beta)$. Let $F$ be a hemicontinuous, monotone angle-bounded map of $X$ into $X^*$ with $0 \in \text{Int}(R(F))$. Let $K$ be a bounded linear map of $L^1(\beta)$ into $L^1(\beta)$ with $\langle Kv, v \rangle \geq 0$ for all $v$ in $L^\infty(\beta)$. Then for each $h$ in $X$, there exists $u$ in $X$ such that $u + KF(u) = h$ and $\langle Kv - KF(u), v - F(u) \rangle \geq 0$ for all $v$ in $L^\infty(\beta)$ with $Kv \in X$.

To prove Theorem 3, we may set $h = 0$ by a change of variables. Let $L_1$ be the mapping from $X^*$ to $X$ with effective domain $L^\infty(\beta)$ and with $L_1(v) = K'(v)$ where $K': L^\infty(\beta) \to L^\infty(\beta)$ is the dual of $K$. Then $L_1$ is monotone and $L_1^*$ is a restriction of $K$. Let $K^#$ be the mapping from $X^*$ to $X$ with domain $D(K^#) = \{v \in L^\infty(\beta) \text{ and } Kv \in X\}$ and $K^#v = Kv$. Since $K^# \subseteq L_1^*$ we may find...
by Theorem 1, a maximal monotone operator $L$ satisfying $K^* \subseteq L \subseteq L_1^*$. Finally one solves $0 \in L^{-1}(u) + F(u)$.

**BIBLIOGRAPHY**


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