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# RESEARCH ANNOUNCEMENTS

## EVERY PLANAR MAP IS FOUR COLORABLE[1]

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The following theorem is proved.

THEOREM. *Every planar map can be colored with at most four colors.*

As has become standard, the four color map problem will be considered in the dual sense as the problem of whether the vertices of every planar graph (without loops) can be colored with at most four colors in such a way that no pair of vertices which lie on a common edge have the same color. The restriction to triangulations with all vertices of degree at least five is a consequence of the work of A. B. Kempe. Over the past 100 years, a number of authors including A. B. Kempe, G. D. Birkhoff, and H. Heesch have developed a theory of reducibility to attack the problem. Simultaneously, a theory of unavoidable sets has been developed and the fusion of these has led to the proof.

A *configuration* is a subgraph of a planar triangulation consisting of a circuit (called the ring) and its interior. A configuration is called *reducible* if it can be shown by certain standard methods that it cannot be immersed in a minimal counterexample to the four color conjecture. (For details, see [3] or [4].) A set of configurations is called *unavoidable* if every planar triangulation contains some member of the set. From the definitions, it is immediate that the four color theorem is proved if an unavoidable set of reducible configurations is provided.

The most efficient known method of producing unavoidable sets of configurations is called the method of discharging. This method treats the planar triangulation as an electrical network with charge assigned to the vertices. Euler's formula is used to show that the initial charge distribution, giving positive charge to vertices of degree five and negative charge to vertices of degree greater than six, has positive total charge. Next, the initial charge is redistributed in a manner which obeys the principle of conservation of charge. This means that some vertices must end up with positive charge. Such an algorithm can be made sufficiently sophisticated that a finite list of neighborhoods of all possible vertices of ultimately positive charge can be described in detail.

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[1]This work appears in full in two papers, *Every planar map is four colorable*; part I, *Discharging*, by K. Appel and W. Haken and part II, *Reducibility*, by K. Appel, W. Haken, and J. Koch. These papers have been submitted to another journal.

Since every triangulation of the plane must contain such a neighborhood, a set $U$ of configurations such that every such neighborhood contains a member of $U$ is clearly unavoidable in planar triangulations.

The major effort in the work was involved in the development of the discharging procedure. Although the authors have made extensive use of computers in the study of discharging algorithms (see [2]), the algorithm employed is technically simpler than their earlier approaches and was implemented by hand. The method actually produces a class of discharging algorithms which differ from one another only in minor details. The particular procedure chosen was determined principally to avoid configurations of ring size greater than fourteen and to employ configurations whose reducibility could be proved without exorbitant use of computer time. The algorithm produced a set $U$ of fewer than 2000 configurations, each of ring size fourteen or smaller.

Reducibility of configurations by computer has been studied by a number of authors, including H. Heesch, S. Gill, and F. Allaire and E. R. Swart. The reducibility check performed in this work involved several computer programs written by John Koch and the authors. The programs used the algorithms of Heesch (as described in [1] and [3]) but were designed to take advantage of the flexibility of the discharging procedure. Thus, rather than employing the most sophisticated known techniques of reducibility they were designed for speed and efficiency in treating those configurations they could prove reducible. Each configuration of the unavoidable set $U$ was shown to be reducible.

The list of configurations used is clearly not a smallest possible such list. It is likely that, by a combination of minor changes in the discharging algorithm, additional detailed analysis and most sophisticated reduction procedures, the number of configurations used in the proof could be reduced by at least twenty five per cent. It seems unlikely, however, that the theorem could be proved by these methods in a way which would avoid massive computations which required the use of computers. This last conclusion is supported by work of E. F. Moore and probabilistic calculations of the authors which indicate that such an argument always requires configurations of ring size fourteen.

## REFERENCES

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3. H. Heesch, *Untersuchungen zum Vierfarbenproblem*, B. I. Hochschulskripten, 810/810a/810b, Bibliographisches Institut, Mannheim, 1969. MR **40** #1303.

4. W. Tutte and H. Whitney, *Kempe chains and the four color problem*, Utilitas Math. **2** (1972), 241–281. MR **46** #8887.

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