Certain models in quantum field theory can be defined by a generalized random process \( \phi(f) = \int \phi(x) f(x) \, dx \) for \( f \in \mathcal{S}(\mathbb{R}^d) \) satisfying the following conditions [3]: (a) Regularity. The expectation of \( e^{\phi(f)} \) is entire analytic on \( \mathcal{S}(\mathbb{R}^d) \); (b) Euclidean invariance (including reflections) of the underlying measure \( d\mu \). This means that

\[
\int \left[ \prod_i \phi(f_i) \right] \, d\mu = \int \left[ \prod_i \phi(\eta f_i) \right] 
\]

Here \( (\eta f)(x) = f(\eta^{-1}x) \) and \( \eta \) belongs to the Euclidean group. This identity induces a unitary transformation \( T_\eta \) on the space \( E = L_2(d\mu) \) of random variables. (c) Reflection positivity. Let \( r \) denote reflection in the \( x_0 \) plane, and let \( v \) be a function of the random variables \( \{\phi(f)\} \) where \( \text{supp} f \) lies in the half space \( x_0 > 0 \). Then

\[
\int \bar{v}(T_r v) \, d\mu \geq 0.
\]

This final condition enables us to define the Hilbert space \( H \) (which plays the role of \( L_2 \) of the state space) and a contraction semigroup \( e^{-tH} \) (which defines the transition probabilities for the process). The inner product on \( H \) is given by (2) after dividing out by the space of null vectors. The semigroup \( e^{-tH} \) arises from translation in the \( x_0 = t \) direction.

The simplest example of a process satisfying the above conditions is the Gaussian process whose generating functional is

\[
\int e^{\psi(f)} \, d\mu_0 = \exp\left(-\frac{1}{2}\langle f, (-\Delta + 1)^{-1}f \rangle\right)L^2.
\]

This process is known as the Ornstein-Uhlenbeck process. For \( d = 1, 2 \) we consider the following limiting process:

\[
\int e^{\phi(f)} \, d\mu^\pm = \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{\int e^{i\phi(f)} \exp(-P_\Lambda) \exp(-Q^2_{R^2 \backslash \Lambda}) \, d\mu_0}{\int \exp(-P_\Lambda) \exp(-Q^2_{R^2 \backslash \Lambda}) \, d\mu_0}
\]


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where
\[ P_\Lambda = \int_\Lambda [\lambda: \phi^4(x) - \phi^2(x) F d(2) x \]
and
\[ Q_\Lambda^\pm = \int_X [\phi^2(x) \mp (4\lambda)^{-\frac{1}{2}}\phi(x)] d(2) x. \]

When \( d = 1 \), \( H = L^2(dx) \) and the generator \( H \) equals \(-\frac{1}{2} d^2/dx - \frac{1}{2} x^2 + \lambda x^4 \).

In this case, the polynomial \( Q^\pm \) has no effect on \( H \) and \( d\mu^\pm = d\mu^- \).

We study the case \( d = 2 \) and show that for small \( \lambda \) the limit exists and depends on \( Q^\pm \). If we set \( Q = 0 \) and replace \( \Delta \) in (3) by the Laplacian with Dirichlet boundary conditions on \( \partial\Lambda \), we show that the translation group \( T_t \) defined on \( E \) does not act ergodically \([1]\), hence the ground state of the corresponding Hamiltonian \( H \) is not unique. The effect of \( Q^\pm \) is to choose a unique ground state.

The dependence on boundary conditions is a phenomenon called phase transition in statistical mechanics and occurs, for example, in the Ising model.

The following results are established in \([2]\).

**Theorem 1.** For small \( \lambda \) and for the quadratic boundary conditions \( Q_\Lambda \) of \([2]\), the limit \( d\mu^\pm \) in (4) exists and defines a measure on \( S(R^2) \) which is ergodic under \( R^2 \)-translation with an exponential mixing rate \( m > 0 \). To define the exponential mixing rate \( m \), let \( A \) and \( B \) be functions of \( \phi(f) \) for \( f \in C^\infty_0(R^2) \).

Then
\[
m = \inf_{A, B} \lim_{|x| \to \infty} \left\{ \frac{\ln(\langle AT_x B \rangle - \langle A \rangle \langle B \rangle)}{|x|} \right\}.
\]

Ergodicity combined with conditions (a)–(c) above are a mild strengthening of the Osterwalder-Schrader axioms. Verification of (a)–(c) gives our next result.

**Theorem 2.** The measure \( d\mu^\pm \) satisfies all Osterwalder-Schrader axioms and has nonzero expectation value
\[
\int \phi(x) d\mu^\pm = \pm (4\lambda)^{-\frac{1}{2}} + O(\lambda^{3/2})
\]
for \( \lambda \ll 1 \).

The above theorem is a nonuniqueness theorem because it implies \( d\mu^+ \neq d\mu^- \). It also proves symmetry breaking, by showing that the symmetry \( \phi \leftrightarrow -\phi \) of the interaction is not a symmetry of the solution \( d\mu^\pm \). The corresponding real time fields satisfy all Wightman axioms.

**Theorem 3.** The measure \( d\mu^\pm \) has moments in the translated variable \( \psi^\pm = \phi \pm (4\lambda)^{-\frac{1}{2}} \) which are \( C^\infty \) as functions of \( \lambda^{\frac{1}{2}} \), at \( \lambda = 0 \), and the derivatives \( \partial/\partial(\lambda^{\frac{1}{2}}) \) in the Taylor’s expansion about \( \lambda = 0 \) can be evaluated explicitly in terms...
of Feynman diagrams. The moments are also analytic in a small sector $|\text{Im } \lambda| \leq \epsilon |\text{Re } \lambda| \ll 1$, $\epsilon \ll 1$.

**Remark.** Theorem 3 makes precise the sense in which $d\mu^2$ is almost Gaussian.

**Bibliography**


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