ON SURFACES OBTAINED FROM QUATERNION ALGEBRAS OVER REAL QUADRATIC FIELDS

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Let $A$ be a totally indefinite division quaternion algebra with center $k = \mathbb{Q}(\sqrt{d})$, $d > 0$, $\mathcal{O}$ a maximal order in $A$, and $\Gamma(1) = \{ \alpha \in \mathcal{O} | \nu(\alpha) = 1 \}$ where $\nu$ is the reduced norm from $A$ to $k$. Fix an isomorphism $\lambda$ such that $A \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$. Then $\lambda(\Gamma(1) \otimes_{\mathbb{Q}} 1) \subseteq SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, and $j(\Gamma(1)) = \Gamma(1)/(\text{center } \Gamma(1))$ acts holomorphically and properly discontinuously on $X = H \times H$, where $H$ is the usual upper half plane. In general, if $\Gamma$ is any group of holomorphic automorphisms of $X$ acting properly discontinuously and without fixed points, then $\Gamma \backslash X$ is a complex manifold. Since $A$ is division the quotient is compact, and it is known to be a projective algebraic variety. In this note we discuss the numerical invariants and second cohomology group of $U(\Gamma) = \Gamma \backslash H \times H$ where $\Gamma$ is commensurable with $\Gamma(1)$.

(A) For any algebraic number field $F$, a quaternion algebra with center $F$ is determined up to isomorphism by a finite set $S(A)$ of prime divisors of $F$. Denote this algebra by $A(F, S(A))$.

THEOREM 1. Assume $h(k) =$ class number of $k = 1$. Let $j(\Gamma(1)) = \Gamma(1)/\{ \pm 1 \}$, $A = A(k, S(A))$, and let

$$\left( \frac{-3}{p} \right)$$

be the Kronecker symbol. $j(\Gamma(1))$ acts on $X$ without fixed points $\Leftrightarrow$ all of the following hold:

\begin{enumerate}
  \item $\left( \frac{-3}{p} \right) = 1$ or $\left( \frac{-D}{p} \right) = 1$
  \item $\left( \frac{-1}{p} \right) = 1$ or $\left( \frac{-D'}{p} \right) = 1$
\end{enumerate}

for some $P \in S(A)$, where $p \mathbb{Z} = P \cap \mathbb{Z}$ and $-D'$ is the discriminant of the field $\mathbb{Q}(\sqrt{-3d})$.

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If \( d = 5 \), \( 3P \in S(A) \) such that \( pZ = P \cap Z \) and \( p \equiv 1 \) (mod 5).

Let \( A^{X++} = \{ \alpha \in A^X \mid \nu(\alpha) \) is totally positive \} and call such \( \alpha \) totally positive. Let \( E^{++} = O^X \cap A^{X++} \). \(|j(E^{++})| = 2\) if \( \epsilon_k \), the fundamental unit of \( k \) greater than 1, is totally positive, and \(|j(E^{++})| = 1\) otherwise.

**Theorem 2.** Assume \( h(k) = 1 \) and \( \epsilon_k \) is totally positive. \( j(E^{++}) \) acts on \( X \) without fixed points if both of the following hold:

1. \( j(\Gamma(1)) \) has no elements of finite order.
2. \( \exists P \in S(A) \) such that \( P \) splits in \( k(\sqrt{-\epsilon_k})/k \).

Consider \( B^{++} = \{ \beta \in A^{X++} \mid \beta \mathcal{O} = 0\beta \} = \text{normalizer of } \Gamma(1) \) in \( A^{X++} \).

If \( h(k) = 1 \) then the class number of a maximal order in \( A \) is also 1. Therefore every 2-sided \( \mathcal{O} \)-ideal is principal. The set of all 2-sided maximal \( \mathcal{O} \)-ideals are in one-to-one correspondence with the prime ideals of \( \mathcal{O}_k \). Let \( P_i = \Pi_i \mathcal{O} \) correspond to \( P_i = \pi_i \mathcal{O}_k \).

**Theorem 3.** Assume \( h(k) = 1 \). Let \( \epsilon \) be a fundamental unit of \( \mathcal{O}_k \). Let \( \{\pi_i\}_{i=1,2,\ldots,n} \) correspond to \( \{\Pi_i \mathcal{O}\}_{i=1,2,\ldots,n} = S(A) \). For these \( \pi_i \) let \( \eta(i_1, i_2, \ldots, i_r) = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_r} \) where \( \pi_{i_s} \neq \pi_{i_t} \) for \( s \neq t \). \( j(B^{++}) \) acts on \( X \) without fixed points if and only if both of the following hold:

1. \( j(E^{++}) \) has no elements of finite order.
2. For all totally positive \( \eta(i_1, i_2, \ldots, i_r) \), \( \exists P \in S(A) \) such that \( P \) splits in \( k(\sqrt{-\eta(i_1, i_2, \ldots, i_r)})/k \), and for all totally positive \( \eta(i_1, i_2, \ldots, i_r)\epsilon \) (for some choice of \( \epsilon \)), \( \exists P \in S(A) \) such that \( P \) splits in \( k(\sqrt{-\eta(i_1, i_2, \ldots, i_r)e})/k \).

(B) Throughout this section \( \Gamma \) is a group commensurable with \( j(\Gamma(1)) \) acting on \( X \) without fixed points. Using a result of Matsushima and Shimura [2] we have

**Proposition 1.** (1) The Euler characteristic \( E \), the geometric genus \( p_g \), and the arithmetic genus \( p_a \) of \( \Gamma \backslash X \) have the following relationship: \( E = 4(p_g + 1) = 4p_a \).

2. The irregularity \( q \) is 0.

3. Then \( m \)th plurigenus \( P_m = (p_g + 1)(2m - 1)^2 \), \( m \geq 2 \).

**Corollary.** \( \Gamma \backslash X \) is a surface of general type.

Using the Riemann-Roch theorem we have

**Corollary.** \( c_1^2 = 8p_g + 8 \), where \( c_1 \) is the first Chern class of \( \Gamma \backslash X \).

Using a formula of Shimizu [4] for the volume of a fundamental domain for the action of \( j(\Gamma(1)) \) on \( X \), and the Gauss-Bonnet theorem we obtain

**Theorem 4.** \( E(U(1)), \) the Euler characteristic of \( j(\Gamma(1)) \backslash X \) is given by
\begin{equation}
E(U(1)) = \frac{B_d}{12} \prod_{P \in S(A)} (N_{k/Q}P - 1)
\end{equation}

where $B_d$ is the generalized Bernoulli number of the numerical character modulo $d$ associated to the field $k = \mathbb{Q}(\sqrt{d})$.

For $d \neq 5$, $B_d$ is an integer. With the aid of a computer, James Maiorana has calculated $B_d$ for $d < 750$.

We have a complete list of surfaces with $p_g = 0$ and $p_g = 1$ which come from groups $\Gamma, j(\Gamma(1)) \subseteq \Gamma \subseteq j(B^+)$.  

c) Let $U(1) = j(\Gamma(1)) \backslash X$ be an algebraic variety. $H_1(U(1), \mathbb{Z})$ is isomorphic to $H^2(U(1), \mathbb{Z})_{\text{torsion}}$ by Poincaré and Pontrjagin duality. Thus

\[ H^2(U(1), \mathbb{Z})_{\text{torsion}} \cong j(\Gamma(1))/[j(\Gamma(1)), j(\Gamma(1))] \cong \Gamma(1)/[\pm 1][\Gamma(1), \Gamma(1)]. \]

By constructing a normal subgroup of $\Gamma(1)$ containing $[\Gamma(1), \Gamma(1)]$, we obtain

**Theorem 5.** Let $j(\Gamma(1))$ act on $X$ without fixed points. Then $|H^2(U(1), \mathbb{Z})_{\text{torsion}}|$ is divisible by $a \cdot b \cdot c \cdot \prod_{P \in S(A)} (N_{k/Q}P + 1)$ where

\[
a = \begin{cases} 
\frac{1}{2} & \text{if } N_{k/Q}P \equiv 1 \pmod{4} \text{ for some } P \in S(A), \\
1 & \text{otherwise}.
\end{cases}
\]

\[
b = \begin{cases} 
4 & \text{if } 3 \text{ } P, Q \text{ such that } P \neq Q, PQ = 2\mathbb{Z} \text{ and } P, Q \notin S(A), \\
2 & \text{if } 3 \text{ } P, Q \text{ such that } PQ = 2\mathbb{Z} \text{ and } P \notin S(A) \text{ but } Q \in S(A), \text{ or if } 3 \text{ } P \text{ such that } P^2 = 2\mathbb{Z} \text{ and } P \notin S(A), \\
1 & \text{otherwise}.
\end{cases}
\]

\[
c = \begin{cases} 
9 & \text{if } 3 \text{ } P, Q \text{ such that } P \neq Q, PQ = 3\mathbb{Z} \text{ and } P, Q \notin S(A), \\
3 & \text{if } 3 \text{ } P, Q \text{ such that } PQ = 3\mathbb{Z} \text{ and } P \notin S(A) \text{ but } Q \in S(A), \text{ or if } 3 \text{ } P \text{ such that } P^2 = 3\mathbb{Z} \text{ and } P \notin S(A), \\
1 & \text{otherwise}.
\end{cases}
\]

**Example.** Let $A = A(\mathbb{Q}(\sqrt{5}, \{P_5, P_{31}\}))$. We have $P_5^2 = 5\mathbb{Z}$, $N_{k/Q}P_5 = 5$, $P_{31}P_{31} = 31\mathbb{Z}$, $N_{k/Q}P_{31} = 31$, $N_{k/Q}P_2 = 4$, $N_{k/Q}P_3 = 9$, $e_k = (1 + \sqrt{5})/2$, $N_{k/Q}e_k = -1$, and $B_5 = 4/5$. $U(1) = j(\Gamma(1)) \backslash X$ is smooth, $E(U(1)) = (1/12) \cdot (4/5)(5 - 1)(31 - 1) = 8$, so $p_g = 1$. $|H^2(U(1), \mathbb{Z})_{\text{torsion}}|$ is divisible by $(1/2)(5 + 1)(31 + 1) = 96$. There are two subgroups between $j(\Gamma(1))$ and $j(B^+)$ yielding $p_g = 0$ surfaces. For more examples see [3].

(D) Let $K$ be the canonical line bundle of a surface of the above type. In conjunction with Gordon Jenkins, we have shown that in the case $P_g = 0$, $3K$ is very ample, that is, $3K$ determines a biholomorphic imbedding into some complex projective space.

Gordon Jenkins [1] has investigated cases where $[k : \mathbb{Q}] \geq 3$. 

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