This note is to announce a characterization of (generalized) path spaces satisfying the Osterwalder-Schrader positivity condition by the associated semigroup, on the lines of the characterization of Markov path spaces by positivity preserving semigroups (e.g. Simon [5], Klein and Landau [3]). In the semigroup characterization Osterwalder-Schrader path spaces are seen to be the natural generalization of Markov path spaces. As an application we discuss the existence of Euclidean fields given a relativistic Wightman field theory.

I. Path spaces and semigroups. A (generalized) path space \((\mathcal{Q}, \Sigma, \mu), \Sigma_0, U(t), R)\) consists of a probability space \((\mathcal{Q}, \Sigma, \mu)\); a distinguished sub-\(\sigma\)-algebra \(\Sigma_0\); a one-parameter group \(U(t)\) of measure preserving automorphisms of \(L_\infty(\mathcal{Q}, \Sigma, \mu)\) which are strongly continuous in measure; a measure preserving automorphism \(R\) of \(L_\infty(\mathcal{Q}, \Sigma, \mu)\) such that \(R^2 = I, RU(t) = U(-t)R,\) and \(RE_0 = E_0R\) where \(E_0\) is the conditional expectation with respect to \(\Sigma_0\); where \(\Sigma\) is generated by \(\bigcup_{t \in R} \Sigma_t, \Sigma_t = U(t) \Sigma_0\). By \(E_+ (E_-)\) we will denote the conditional expectation with respect to \(\Sigma_+ (\Sigma_-)\), the \(\sigma\)-algebra generated by \(\bigcup_{t \geq 0} \Sigma_t (\bigcup_{t \leq 0} \Sigma_t)\). The path space is said to be Osterwalder-Schrader if \(\langle RF, F \rangle \geq 0\) for every \(F \in L_2(\mathcal{Q}, \Sigma_+, \mu)\). It is said to be Markov if \(RE_0 = E_0\) and \(E_+ E_- = E_+ E_0 E_-\).

Every Markov path space is Osterwalder-Schrader [4]. In the case of a Markov path space \(P(t) = E_0 U(t) E_0\) gives a positivity preserving semigroup on \(L_2(\mathcal{Q}, \Sigma_0, \mu)\) [5], [3]. Given an Osterwalder-Schrader path space there exists [4] a Hilbert space \(H\) and a contraction \(V: L_2(\mathcal{Q}, \Sigma_+, \mu) \rightarrow H\) such that \(V\) has dense range and \(P(t)V(F) = V(U(t)F)\) for \(F \in L_2(\mathcal{Q}, \Sigma_+, \mu)\) and \(t \geq 0\) defines a strongly continuous selfadjoint contraction semigroup on \(H\). If \(\Omega = V(1)\), then \(\|\Omega\| = 1\) and \(P(t)\Omega = \Omega\) for all \(t \geq 0\).

For Osterwalder-Schrader path spaces we must look at another piece of structure, which is hidden in the Markov case.

Lemma. Let \(((\mathcal{Q}, \Sigma, \mu), \Sigma_0, U(t), R)\) be an Osterwalder-Schrader path...
space, and let \( H, V, P(t), \Omega \) be as above. Then, if \( f \in L_\infty(Q, \Sigma_0, \mu) \), \( \widetilde{V}(F) = V(f F) \) for \( F \in L_2(Q, \Sigma_+, \mu) \) defines a bounded operator on \( H \) with \( \|f\| = \|f\|_\infty \), and \( \mathfrak{A} = \{ \widetilde{f} f^* \in L_\infty(Q, \Sigma_0, \mu) \} \) is a commutative von Neumann algebra of operators on \( H \), with \( \Omega \) as a separating vector. Moreover, for any \( t_1 \leq t_2 \leq \cdots \leq t_n, f_{t_i} = U(t_i) f_i \) where \( f_i \in L_\infty(Q, \Sigma_0, \mu) \) and \( i = 1, 2, \ldots, n \),

\[
\int f_{t_1} f_{t_2} \cdots f_{t_n} \, d\mu = \langle \Omega, \widetilde{f}_1 P(t_2 - t_1) \widetilde{f}_2 \cdots P(t_n - t_{n-1}) \widetilde{f}_n \Omega \rangle.
\]

\((H, P(t), \mathfrak{A}, \Omega)\) is called the associated semigroup structure. If \(((Q, \Sigma_0, \mu), \Sigma_0, U(t), R)\) is a Markov path space, \((L_2(Q, \Sigma_0, \mu), E_0 U(t) E_0, L_\infty(Q, \Sigma_0, \mu), 1)\) is its associated semigroup structure.

**Definition.** A positive semigroup structure \((H, \mathfrak{A}, \Omega, \Omega)\) consists of a Hilbert space \( H \); a strongly continuous selfadjoint contraction semigroup \( P(t) \) on \( H \); a commutative von Neumann algebra \( \mathfrak{A} \) of operators on \( H \); a unit vector \( \Omega \in H \); such that \( P(t)\Omega = \Omega \) for all \( t \geq 0 \); \( \Omega \) is a cyclic vector for the algebra generated by \( \mathfrak{A} \cup \{ P(t) \mid t \geq 0 \} \), i.e. the linear span of \( \{ P(t_1) f_1 P(t_2) \cdots P(t_n) f_n \Omega \mid f_1, \ldots, f_n \in \mathfrak{A}, t_1, \ldots, t_n \geq 0 \} \) is dense in \( H \); and for all \( f_1, \ldots, f_n \in \mathfrak{A}^+ = \{ f \in \mathfrak{A} \mid f \geq 0 \} \) and \( t_1, \ldots, t_n \geq 0 \), \( \langle \Omega, P(t_1) f_1 P(t_2) \cdots P(t_n) f_n \Omega \rangle \geq 0 \).

Osterwalder-Schrader path spaces are characterized by positive semigroup structures.

**Theorem.** Let \(((Q, \Sigma, \mu), \Sigma_0, U(t), R)\) be an Osterwalder-Schrader path space and \((H, P(t), \mathfrak{A}, \Omega)\) its associated semigroup structure. Then \((H, P(t), \mathfrak{A}, \Omega)\) forms a positive semigroup structure.

Conversely, let \((H, P(t), \mathfrak{A}, \Omega)\) be a positive semigroup structure. Then there exists an Osterwalder-Schrader path space such that \((H, P(t), \mathfrak{A}, \Omega)\) is its associated semigroup structure.

**Corollary.** Let \(((Q, \Sigma, \mu), \Sigma_0, U(t), R)\) be an Osterwalder-Schrader path space, and \((H, P(t), \mathfrak{A}, \Omega)\) its associated semigroup structure. The path space is Markov if and only if \( \Omega \) is a cyclic vector for \( \mathfrak{A} \).

The details will appear elsewhere [2].

**II. Existence of Euclidean fields.** Our Theorem can be used to construct Euclidean fields given a relativistic Wightman field theory, in the same way Simon ([5], [6, Chapter IV]) used the similar result for Markov path spaces and positivity preserving semigroups to construct Euclidean fields. In Simon's scheme Axioms (S3) and (S4) [6, p. 120] are the basic elements in the construction of Euclidean fields. We can replace these axioms by the weaker:

**Axiom 3'.** The von Neumann algebra \( \mathfrak{A} \) generated by the time zero fields is abelian; and the vacuum \( \Omega \) is a cyclic vector for the von Neumann algebra generated by the fields at all fixed times.
AXIOM 4'. For all $F_1, \ldots, F_n \in \mathfrak{A}^+ = \{ F \in \mathfrak{A} | F \geq 0 \}$ and $t_1, \ldots, t_n \geq 0$, $\langle \Omega, e^{-t_1 H F_1}e^{-t_2 H F_2} \cdots e^{-t_n H F_n} \Omega \rangle \geq 0$.

We can then construct Euclidean fields satisfying Nelson's axioms, except for the Markov property which is replaced by the Osterwalder-Schrader positivity condition.

A detailed version of our axiom scheme will appear elsewhere [1], [2].

REFERENCES

1. A. Klein, When do Euclidean fields exist?, Letters in Mathematical Physics, 1 (1976), 131-133.


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