1. Introduction. For definitions and notation in what follows, see [4] and [5]. If $A$ is an infinite set and $\varphi(y_1, \ldots, y_n, R, Y_1, \ldots, Y_m) = \varphi(\vec{y}, R, \vec{Y})$ is a second order relation on $A$, we call $\varphi$ operative if $R$ is n-ary. For such a $\varphi$ let

$$I^t_\varphi = \bigcup_{\eta < t} I^t_\varphi \left\{ \varphi(\vec{y}, \vec{Y}) : \varphi(y, \vec{Y} : (\vec{y}, \vec{Y}) \in \bigcup_{\eta < t} I^t_\varphi, \vec{Y} \right\}$$

and $I_\varphi = \bigcup I^t_\varphi$.

If $F$ is a collection of second order relations (for simplicity collection of operators) on $A$, then $F$-IND$^2$ is the class of all second order relations of the form $\psi(\vec{x}, \vec{Y}) \equiv I_\varphi(\vec{a}, \vec{x}, \vec{Y})$, for some operative $\varphi(\vec{u}, \vec{x}, R, \vec{Y})$ in $F$ and constants $\vec{a}$ from $A$. As in [5] $F$-IND is the class of all relations on $A$ which are in $F$-IND$^2$. We let $F_{\text{mon}}$ be the collection of all operative $\varphi(\vec{y}, R, \vec{Y})$ in $F$ which are monotone on $R$ and we put $\neg F = \{ \neg \varphi : \varphi \in F \}$. A collection of operators $F$ on $A$ is adequate if it contains all the $\Pi^0_1(C)$ second order relations, where $C$ is a coding scheme on $A$ and is closed under $\land$, $\lor$, $\exists A$ and trivial combinatorial substitutions. Let $WF(S) \leftrightarrow S$ be a well-founded relation on $A \leftrightarrow \exists 1 \forall(a_0, a_1, \ldots, a_i) \in S$.

**Theorem 1.** Let $F$ be an adequate collection of operators on an infinite set $A$. If $WF(\neg F \land \exists F \subseteq F_{\text{mon}}$-IND$^2$, then $F$-IND$^2 = F_{\text{mon}}$-IND$^2$.

2. Elementary induction. Let $EL$ be the collection of all the elementary second order relations on a structure $A = (A, R_1, \ldots, R_r)$ and let $EL^+$ be the subcollection of $EL_{\text{mon}}$ consisting of all operative $\varphi(\vec{x}, R, \vec{Y})$ which are definable by positive in $R$ elementary formulas. One usually writes $EL^+$-IND$^2$ = IND$^2$ and $EL^+$-IND = IND. Clearly IND$^2 \subseteq EL_{\text{mon}}$-IND$^2 \subseteq EL$-IND$^2$ and it is well known that IND$^2$ is a tiny part of $EL$-IND$^2$ for (say) almost acceptable $A$'s. By a basic result of Kleene and Spector for $\omega$ and Barwise-Gandy-Moschovakis in general (see [4, §8A]), on every countable almost acceptable structure, IND$^2 = EL_{\text{mon}}$-IND$^2 = \Pi^1_1$. On the other hand, letting $WF^n(S) \leftrightarrow S$ is a 2n-ary relation on $A$ which is well founded (viewed as binary on $A^n$), we have

**Corollary 1.** Let $A$ be an infinite structure such that each $WF^n$ is elementary. Then $EL_{\text{mon}}$-IND$^2 = EL$-IND$^2$.

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A more detailed level-by-level version of Corollary 1 is the following, where we just write $\Sigma^0_m$, $\Pi^0_m$ instead of $\Sigma^0_m(C)$, $\Pi^0_m(C)$, where $C$ is a hyperelementary coding scheme on $A$.

**Corollary 2.** Let $A$ be an almost acceptable structure. If $m \geq 2$ and $WF \in \Pi^0_m$, then for all $n \geq m$, $\Sigma^0_n$-IND$^2 = (\Sigma^0_n)^{\text{mon}}$-IND$^2$.

So, for example, in the structure of analysis $\mathbb{R}$ this says that $\Sigma^1_n$ monotone operators on $\mathbb{R}$ inductively define the same relations as arbitrary $\Sigma^1_n$ operators, when $n \geq 2$. Similarly for $\Sigma^1_1$. The following rather curious result can be also established by the methods used to prove Theorem 1. If $A = \langle A, R_1 \cdots R_i \rangle$ is a structure, by an elementary quantifier $Q$ on $A$ we understand a quantifier on $A$ which viewed as a second-order relation is elementary.

**Theorem 2.** Let $A$ be an acceptable structure in which $WF$ is elementary. There is an elementary quantifier $Q$ on $A$ such that for every inductive relation $R$ on $A$, there is an inductive relation $R^*$ on $A$ such that $\forall R(x) \leftrightarrow QyR^*(x, y)$.

This should be compared with a result of Moschovakis [3] in higher type recursion, where “inductive” is replaced by “semirecursive in a total object of type $\geq 3$” and $Q$ becomes the existential quantifier (on an appropriate space).

**Remarks.** (i) We conjecture that in Theorem 1 (and correspondingly in Corollary 1) the hypothesis $WF \in \forall \forall F$ can be weakened to $WF \in \forall (F^{\text{mon}}$-IND$^2)$. (ii) In a direction opposite to that of Corollary 1 one has the following theorem of Nyberg (unpublished): Let $A$ be almost acceptable. If $\text{IND} \subseteq (\text{EL}^{\text{mon}}$-IND), then $\text{EL}^{\text{mon}}$-IND = IND. Thus for most structures occurring in practice, $\text{EL}^{\text{mon}}$-IND is either IND or EL-IND.

3. Further corollaries and applications to Spector classes. An immediate consequence of Theorem 1 is also the following result of Harrington and Moschovakis [2]. (Given a structure $A$ and a quantifier $Q$ on $A$ we abbreviate by $Q$-IND the class of second order relations which are positive $L^A(Q)$-inductive (see [4, p. 49]).

**Corollary 3.** (Harrington-Moschovakis [2]). Let $A$ be an almost acceptable structure and let $Q$ be a quantifier on $A$. If $F = \forall (Q$-IND$^2)$, then $F$-IND$^2 = F^{\text{mon}}$-IND$^2$.

This generalizes a result of Grilliot to the effect that over $\omega$, $\Sigma^1_1$-IND$^2 = (\Sigma^1_1)^{\text{mon}}$-IND$^2$. The original proof of Corollary 2 in [2] yields the stronger statement that for $F = \forall (Q$-IND$^2)$, $F$-IND$^2 = F^{\text{pos}}$-IND$^2$ and also shows that $F$-IND$^2 = Q^+$-IND$^2$, where $Q^+$ is the next quantifier of $Q$ (see [1]). Turning now to Spector classes we can obtain the following, where the notions involved are explained in [5].

**Theorem 3.** Let $\Gamma$ be a Spector class on $A$, and let $F$ be a reasonable,
nonmonotone class of operators on $A$ closed under $\exists^A$. If $WF \subseteq \Gamma$, then $\Gamma$ is $F$-compact iff $\Gamma$ is $F^\text{mon}_*$-compact, where $F^\text{mon}_* = \{\varphi(R) : \varphi \in F, \varphi \text{ monotone}\}$. In particular if $F$ is typical, nonmonotone, $F^\text{mon}_*$-IND is a Spector class iff $F^\text{mon}_*$-IND = $F$-IND.

Further applications of the methods developed here to the theory of "second order" Spector classes as well as details and proofs of the results announced here will appear elsewhere.

REFERENCES


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