THE RATIONAL HOMOTOPY OF FIXED POINT SETS
OF TORUS ACTIONS

BY CHRISTOPHER ALLDAY

Communicated by E. H. Brown, Jr., July 20, 1976

1. Introduction. Let \( X \) be a connected topological space, whose Sullivan-de Rham minimal model, \( M(X) \), is finitely generated. Following Halperin [8], we shall denote the indecomposable quotient of \( M(X) \) by \( \Pi^\psi(X) \), and call it the pseudo-dual rational homotopy of \( X \). If \( X \) is simply-connected, then \( \Pi^\psi(X) \) is naturally isomorphic to \((\pi_n(X) \otimes \mathbb{Q})^*\), for all \( n \geq 1 \). (See [4] and [8] for detailed treatment of \( \Pi^\psi(X) \).)

DEFINITION 1.1. If \( \dim \Pi^\psi(X) < \infty \), then we shall say that \( X \) has finite dimensional rational homotopy (FDRH), and we shall define the Euler-Poincaré homotopy characteristic of \( X \) to be \( \chi(X) = \sum_{n=1}^{\infty} (-1)^n \dim \Pi^\psi_n(X) \).

In this note we announce some results, which relate \( \Pi^\psi(X) \) to \( \Pi^\psi(F) \), where \( F \) is a component of the fixed point set of a torus group action on \( X \). Further results and detailed proofs will appear in [2] and [3].

2. Results. Although more general conditions would suffice, we shall assume, for simplicity, throughout this section, that \( X \) is a compact topological manifold, that a torus \( T \) is acting on \( X \) locally smoothly (that is, with linear slices), and that the fixed point set, \( X^T \), is nonempty. Our first theorem is the following.

THEOREM 2.1. If \( X \) has FDRH, and if \( F \) is a component of \( X^T \), then \( F \) has FDRH, and \( \chi(F) = \chi(X) \). Furthermore,

\[
\sum_{n=1}^{\infty} \dim_q \Pi^\psi_n(F) \leq \sum_{n=1}^{\infty} \dim_q \Pi^\psi_n(X);
\]

and

\[
\sum_{n=0}^{\infty} \dim_q \Pi^\psi_{n+1}(F) \leq \sum_{n=0}^{\infty} \dim_q \Pi^\psi_{n+1}(X).
\]

We also have the following generalization of Bredon’s inequalities [5].

THEOREM 2.2. If \( X \) has FDRH, then, for all \( n \geq 1 \),

\[
\dim_q \Pi^\psi_n(F) \leq \sum_{k=0}^{\infty} \dim_q \Pi^\psi_{n+2k}(X).
\]
Our third theorem is a generalized Golber formula ([1], [6], [7] and [9]). We shall assume now that $X$ has FDRH, and that $\Pi^2_n(X) = 0$, for all $n \geq 1$. It follows that $X^K$ is connected, for any subtorus $K \subseteq T$. From Theorem 2.1 it follows also that $\Pi^2_n(X^K) = 0$, for all $n \geq 1$, and that $X^K$ has FDRH. With this in mind we make the following definition.

**Definition 2.3.** Suppose that $\Pi^*_n(X^K)$ has a basis (as a rational vector space) of elements with degrees $\alpha_i(K)$, $1 \leq i \leq s$.

Set

$$e(K) = \prod_{1 \leq i < j \leq s} (\alpha_i(K) + 1)(\alpha_j(K) + 1).$$

If $K = \{e\}$, so that $X^K = X$, then set $e(K) = e(X)$.

The generalized Golber formula is as follows.

**Theorem 2.4.**

$$e(X) - e(T) - \sum_H [e(H) - e(T)] = \sum_K \left[ e(K) - e(T) - \sum_{H \supset K} \{e(H) - e(T)\} \right],$$

where $\Sigma_H$ runs over all subtori of $T$ of corank one, $\Sigma_K$ runs over all subtori of $T$ of corank two, and $\Sigma_{H \supset K}$ runs over all subtori of $T$ of corank one, which contain $K$.

In [3], we obtain further formulae of this kind, and give a general solution to Problem 9 of [9, p. 148].

3. **Method of proof.** The following theorem is the main technical device which we use.

**Theorem 3.1.** If $S$ is a commutative overring of the rational numbers, and if $A_S$ is the category of differential $(\mathbb{Z}/2\mathbb{Z})$-graded algebras over $S$ (with $S$ having degree 0), then $A_S$ is a closed model category.

The proof of this theorem is a straightforward analogue of the proof of Theorem 4.3 of [4].

Theorem 3.1 allows us to reproduce a localization-cum-ideal theory for $\Pi^*_n$, analogous to that for equivariant cohomology produced by Chang and Skjelbred [6].

**REFERENCES**


2. ———, *On the rational homotopy of fixed point sets of torus actions* (to appear).


