PERTURBATION AND ANALYTIC CONTINUATION
OF GROUP REPRESENTATIONS

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Communicated by C. Davis, July 6, 1976

ABSTRACT. I introduce a theory of noncommutative bounded perturbations of Lie algebras of unbounded operators. When applied to group representations, it leads to an analytic embedding of the dual object of some semisimple Lie groups into the bounded operators on corresponding Hilbert spaces of $K$-finite vectors.

1. Introduction. I announce a general theorem on analytic continuation of group representations which is based on perturbation theory for linear operators. This result is a contribution of the author to a series of joint results with R. T. Moore reported in detail in [3]. Applications of the theorem to quasi-simple Banach representations of $SL(2, \mathbb{R})$, due to Moore, will be announced separately by him. The theorem introduces a perturbation theory for representations of Lie groups which generalizes the classical perturbation theory (due to R. S. Phillips [2, p. 389]) for one-parameter (semi) groups of bounded linear operators on a Banach space $E$. Let $\{\pi(t): -\infty < t < \infty\}$ be such a strongly continuous one-parameter group ($C_0$ group) acting on a Banach space $E$. Let $A$ be the infinitesimal generator of $\pi$, and let $U$ be a "small" (bounded, say) perturbation of $A$, $B = A + U$. Then $B$ generates a $C_0$ group $\{\pi(t)\}$ on $E$, and this group depends analytically on $U$ (in a sense which is specified in [2, p. 404]). In my theorem the real line $\mathbb{R}$ is replaced by a Lie group $G$, and $A$ is replaced by a Lie algebra $L$ of unbounded operators in $E$. $U$ is going to be a tuple $(U_1, \ldots, U_r)$ of bounded operators. In that way I obtain a surprisingly simple analytic continuation picture for a wide class of induced representations, and other unitary and nonunitary representations.

2. Assumptions. I first restrict the class of perturbations $U$ to be considered. In order to make sure that $\pi_U$ is a representation of the same group for all $U$, I assume that the corresponding infinitesimal operator Lie algebras $L_U$ are all algebraically isomorphic.

Let $D$ be a linear space. Let $\mathfrak{U}(D)$ be the algebra of linear endomorphisms of $D$. It is also a real Lie algebra when equipped with the commutator bracket, $[A, B] = AB - BA$ for $A, B \in \mathfrak{U}(D)$. The Lie algebra $L$ generated by a subset $S$ of $\mathfrak{U}(D)$ is defined to be the smallest real Lie subalgebra of $\mathfrak{U}(D)$ which contains $S$. 


Sponsored by Odense University, Denmark.
S. (The elements in L are real linear combinations of elements in S and commutators, possibly iterated, of such elements.)

Let $L_0$ be a finite-dimensional Lie subalgebra of $\mathfrak{L}(D)$, and let $A = (A_1, \cdots, A_r)$ be a basis for $L_0$. The order of the operators $A_k$ is essential only when addition of operator tuples is performed: $A + U = B$ with $B_k = A_k + U_k$ and $U_k \in \mathfrak{L}(D)$ for $1 \leq k \leq r$. I consider $r$-tuples $U$ with the property that the Lie algebra $L_U$ generated by $A + U$ is algebraically isomorphic to some fixed finite-dimensional Lie algebra $\mathfrak{g}$ for all $U$: $L_U \approx \mathfrak{g}$. The set of such $r$-tuples is denoted by $\mathcal{U}$. Then there are simply connected Lie groups $G$ (resp. $G_0$) with Lie algebras $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) such that $L_0 \approx \mathfrak{g}_0$.

I restrict the class $\mathcal{U}$ of perturbations further. Let $\| \cdot \|$ be a fixed norm on $D$. Put $\| x \|_0 = \| x \|$ and $\| x \|_n = \max\{ \| A_{i_1} \cdots A_{i_n} x \| : 0 \leq i_k \leq r \}$ for $x \in D$ and $n = 1, 2, \ldots$. (Define $A_0$ to be the identity $I$ on $D$.) An element $V \in \mathfrak{L}(D)$ is said to be $\| \cdot \|_n$-bounded if there is a finite constant $c_n$ such that $\| Vx \|_n \leq c_n \| x \|_n$ for all $x \in D$. Let $D_n$ be the completion of $(D, \| \cdot \|_n)$ for $n = 0, 1, \ldots$. Put $D_0 = E$. I assume that the operators $A_k$ are closable when viewed as unbounded operators in $E$. Hence $D_1 \subset E$ (cf. [3]). If $V$ is $\| \cdot \|_0$-bounded, it extends to a bounded operator on $E$, $V \in L(E)$. Let $V \in \mathfrak{L}(D)$ be $\| \cdot \|_0$-bounded. Then $V$ is $\| \cdot \|_n$-bounded for given $n$ if the commutators $[A_{i_1}, [A_{i_2}, \ldots, [A_{i_n}, V] \ldots ]]$ are $\| \cdot \|_0$-bounded for all $i \leq i_k \leq r$.

Consider the following subset $\mathcal{V}$ of $\mathcal{U}$: $U = (U_1, \ldots, U_r)$ belongs to $\mathcal{V}$ if and only if each $U_k$ is $\| \cdot \|_0$-bounded and one of the following two conditions is satisfied:

(i) each $U_k$ is $\| \cdot \|_n$-bounded for all $n$; or

(ii) $\{ A_k + U_k \}$ is a basis for $L_U$ and each $U_k$ is $\| \cdot \|_1$-bounded.

3. The Theorem. Let $L_0 \subset \mathfrak{L}(D)$ be an operator Lie algebra, $A = (A_1, \cdots, A_r)$ a basis for $L_0$, and let the class $\mathcal{V}$ of bounded perturbations be as described above. Suppose that $L_0$ exponentiates to a $C_0$ representation $\pi$ of $G_0$ on $E$.

(a) Then $L_U$ exponentiates to a $C_0$ representation of $G$ on $E$ for all $U \in \mathcal{V}$. We denote the exponential by $\pi_U$.

(b) Let $\Omega$ be a complex domain (in one or several dimensions). Let $z \rightarrow U(z) = (U_1(z), \ldots, U_r(z))$ be an analytic function which is defined on $\Omega$ and has its range in $\mathcal{V}$. Then $\pi_{U(z)}$ is analytic as a function of $z$, i.e., $z \rightarrow \pi_{U(z)}(g)$ is analytic for all $g \in G$.

(c) The representations $\pi$ and $\pi_U$ have the same space of $C^\infty$-vectors for all $U \in \mathcal{V}$.

Remark. A suitable class of analytic perturbations $U$ gives representations $\pi_U$ which have the same space of analytic vectors as $\pi$.

The proof is based on two exponentiation theorems due to the co-authors of
I state those theorems as lemmas here. They are significant improvements of results announced in [4], and appear below for the first time in their strengthened form.

**Lemma 1.** Let $D$ be a normed linear space, and $E$ the corresponding completion. Let $L \subset \mathcal{U}(D)$ be a finite-dimensional Lie algebra. Suppose $L$ is generated (in the Lie sense) by a subset $S$ such that every $A \in S$ is closable and the closure $\overline{A}$ generates a $C_0$ group $\{ \pi(t, A) : t \in \mathbb{R} \} \subset L(E)$.

If $D$ is invariant under $\pi(t, A)$ for $t \in \mathbb{R}$ and $A \in S$, and $t \to B\pi(t, A)x$ is locally bounded for all $B, A \in S$ and $x \in D$, then $L$ exponentiates.

**Lemma 2.** Let $L$ and $S$ be as above. (This means that we have $C_0$ groups $\{ \pi(t, A) \}$ for $A \in S$, and there are finite constants $\omega_A$ such that

$$
\sup_t e^{-|t|\omega_A} \| \pi(t, A) \| < \infty.
$$

Let $B_1, \ldots, B_d$ be a basis for $L$. Put $B_0 = I$, and

$$
\|x\|_1 = \max \{ \|B_i x\| : 0 \leq i \leq d \}
$$

for $x \in D$.

Suppose each $A \in S$ satisfies the condition: (GD) There are complex numbers $\lambda_{\pm}$ such that $\text{Re } \lambda_+ > \omega_A + |\text{ad } A|$, $\text{Re } \lambda_- < -\omega_A - |\text{ad } A|$, and the ranges of $\lambda_{\pm} I - A$ are $\| \cdot \|_1$-dense in $D$. Then $L$ exponentiates.

**Proof Sketch (a).** Suppose $L_0$ exponentiates to a representation $\pi$. Let $U \in \mathcal{U}$, and suppose that (i) holds. Then one may apply bounded Phillips perturbations to each of the spaces $D_n = D_n(\pi)$ (cf. [1, Proposition 1.1]) and conclude that each $\overline{B}_k = \overline{A}_k + \overline{U}_k$ generates a $C_0$ group $\pi(t, B_k)$ which leaves $D_\infty$ invariant. So Lemma 1 applies to $L_U$, with $D$ replaced by $D_\infty$.

If (ii) holds, then apply Lemma 2 to $L_U$. Bounded Phillips perturbation in $D_1$ shows that $\pi(t, B_k)$ restricts to a $C_0$ group in $L(D_1)$. Condition (GD) is a simple consequence of this.

**Remark.** The lemmas are hard to apply directly to operator Lie algebras that arise in applications. Fortunately many of these can be shown to be perturbations of a base-point Lie algebra to which the lemmas easily apply.

At this point I have verified, using the theorem, that the dual $\hat{G}$ of the 3- or the 15-dimensional conformal group is analytically embedded via $\pi_U \rightarrow U$ in $\mathcal{B}(H)$ for a common Hilbert space $H$. The range consists of operators which are linear combinations of bounded shifts modulo the compacts (and occasionally Hilbert-Schmidt). This gives new and simple metrics on $\hat{G}$, and thus realizes ideas that were recently suggested to me by Professor I. E. Segal.
REFERENCES


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