STATISTICAL INDEPENDENCE OF LINEAR CONGRUENTIAL
PSEUDO-RANDOM NUMBERS

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Given a modulus $m > 2$ and a multiplier $\lambda$ relatively prime to $m$, a sequence $y_0, y_1, \ldots$ of integers in the least residue system mod $m$ is generated by the recursion $y_{n+1} \equiv \lambda y_n \pmod{m}$ for $n = 0, 1, \ldots$, where the initial value $y_0$ is relatively prime to $m$. The sequence $x_0, x_1, \ldots$ in the interval $[0, 1)$, defined by $x_n = y_n/m$ for $n = 0, 1, \ldots$, is then a sequence of pseudo-random numbers generated by the linear congruential method. The sequence is periodic, with the least period $\tau$ being the exponent to which $\lambda$ belongs mod $m$.

For fixed $s > 2$, consider the $s$-tuples $x_n = (x_n, x_{n+1}, \ldots, x_{n+s-1})$, $n = 0, 1, \ldots$. We determine the empirical distribution of the $s$-tuples $x_0, x_1, \ldots$ and compare it with the uniform distribution on $[0, 1]^s$. The original sequence $x_0, x_1, \ldots$ of linear congruential pseudo-random numbers passes the serial test (for the given value of $s$) if the deviation between these two distributions is small. To measure this deviation, we introduce the quantity

$$D_N = \sup_J |F_N(J) - V(J)| \quad \text{for } N \geq 1,$$

where the supremum is extended over all subintervals $J$ of $[0, 1]^s$, $F_N(J)$ is $N^{-1}$ multiplied by the number of terms among $x_0, x_1, \ldots, x_N$ falling into $J$, and $V(J)$ denotes the volume of $J$.

For a nonzero lattice point $h = (h_1, \ldots, h_s) \in \mathbb{Z}^s$, let $r(h)$ be the absolute value of the product of all nonzero coordinates of $h$. We set

$$R(s)(\lambda, m, q) = \sum_{h \pmod{m} : h \cdot \lambda \equiv 0(q)} (r(h))^{-1},$$

where the sum is extended over all nonzero lattice points $h$ with $-m/2 < h_j \leq m/2$ for $1 \leq j \leq s$ and $h \cdot \lambda = h_1 + h_2\lambda + \cdots + h_s\lambda^{s-1} \equiv 0 \pmod{q}$. For prime moduli $m$, a somewhat simplified version of our result reads as follows.

**Theorem 1.** For a prime $m$ and for a multiplier $\lambda$ belonging to the exponent $\tau$ mod $m$, we have

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\[ D_\tau < \frac{s}{m} + \min \left( 1, \frac{\sqrt{m - r}}{\tau} \right) \left( \frac{2}{\pi} \log m + \frac{7}{5} \right)^s + \frac{1}{2} R^{(s)}(\lambda, m, m). \]

The second term in the upper bound is nonincreasing as a function of \( \tau \) and so becomes minimal for \( \tau = m - 1 \). Values of \( \lambda \) that minimize \( R^{(s)}(\lambda, m, m) \) are of fundamental importance in the theory of good lattice points in the sense of Korobov and Hlawka (see [2, Chapter 2, §5]). We conclude that a multiplier \( \lambda \) is favorable with regard to the \( s \)-dimensional serial test if \( \lambda = (1, \lambda, \ldots, \lambda^{s-1}) \) is a good lattice point mod \( m \) (or, equivalently, \( \lambda \) is an optimal coefficient mod \( m \)) and \( \lambda \) is a primitive root mod \( m \). It can be shown that there exist primitive roots \( \lambda_0 \) mod \( m \) for which \( R^{(s)}(\lambda_0, m, m) \) is of the order of magnitude \( m^{-1} \log^s m \log \log m \).

For an odd prime power \( m = p^\alpha, p \) prime, \( \alpha \geq 2 \), and for \( |\lambda| > 1 \), let \( \tau(p) \) be the exponent to which \( \lambda \) belongs mod \( p \) and let \( \beta \) be the largest integer such that \( p^\beta \) divides \( \lambda^{\tau(p)} - 1 \).

**Theorem 2.** For an odd prime power modulus \( m = p^\alpha \) with \( \alpha > \beta \), we have

\[ D_\tau < \frac{s}{m} + \frac{1}{2} R^{(s)}(\lambda, m, p^{\alpha-\beta}). \]

**Theorem 3.** If \( m = 2^\alpha \) with \( \alpha \geq 3 \) and \( \lambda \equiv 5 \pmod{8} \), then

\[ D_\tau < \frac{s}{m} + \frac{1}{2} R^{(s)}(\lambda, m, 2^{\alpha-2}). \]

If \( m = 2^\alpha \) with \( \alpha \geq 4 \) and \( \lambda \equiv 3 \pmod{8} \), then

\[ D_\tau < \frac{s}{m} + \frac{1}{2} R^{(s)}(\lambda, m, 2^{\alpha-1}) + \frac{1}{2\sqrt{2}} (R^{(s)}(\lambda, m, 2^{\alpha-3}) - R^{(s)}(\lambda, m, 2^{\alpha-2})). \]

Since the upper bounds in Theorems 2 and 3 can be estimated in terms of \( R^{(s)}(\lambda, m', m') \) with a suitable \( m' < m \), the remarks following Theorem 1 apply, *mutatis mutandis*, to prime power moduli.

For computational purposes, it is more convenient to replace \( R^{(s)}(\lambda, m, m) \) by the quantity

\[ \rho^{(s)}(\lambda, m) = \min_h r(h), \]

where the minimum is extended over the range of lattice points used in the definition of \( R^{(s)}(\lambda, m, m) \).

**Theorem 4.** For any dimension \( s \geq 2 \) and for any integers \( m \geq 2 \) and \( \lambda \), we have

\[ R^{(s)}(\lambda, m, m) < \rho^{-1}(\log 2)^{1-s}((2 \log m)^s + 4(2 \log m)^{s-1}) \]

\[ + \rho^{-1}2^{s+1}(2^{s-2} - 1) \left( \frac{k + s - 2}{s - 1} \right), \]

where \( \rho = \rho^{(s)}(\lambda, m) \) and \( k = \lfloor \log m/\log 2 \rfloor \).
There exists an interesting relationship between the two-dimensional serial test and continued fractions. It is based on the fact that $R^{(2)}(\lambda, m, m)$ can be estimated in terms of the partial quotients in the expansion of $\lambda/m$ into a finite simple continued fraction. As a consequence, one obtains that $\lambda$ is favorable with regard to the distribution of pairs whenever these partial quotients are small. This is in accordance with results of Dieter [1] concerning the case $s = 2$.

The proofs of Theorems 1, 2 and 3 depend on estimates for exponential sums with linear recurring arguments established in [3]. The case of inhomogeneous linear congruential pseudo-random numbers and the serial test for parts of the period can be treated by similar techniques (see [5]).

Details and proofs, as well as further results, will appear in [4].

REFERENCES

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