because calculus is about the real numbers. The book offers no evidence that the hyperreal numbers are anything except a device for proving theorems about the real numbers. They are not even an efficient device, depending as they do on axioms V* and VI*, among other things.

The technical complications introduced by Keisler's approach are of minor importance. The real damage lies in his obfuscation and devitalization of those wonderful ideas. No invocation of Newton and Leibniz is going to justify developing calculus using axioms V* and VI*–on the grounds that the usual definition of a limit is too complicated!

Although it seems to be futile, I always tell my calculus students that mathematics is not esoteric: It is common sense. (Even the notorious ε, δ definition of limit is common sense, and moreover is central to the important practical problems of approximation and estimation.) They do not believe me. In fact the idea makes them uncomfortable because it contradicts their previous experience. Now we have a calculus text that can be used to confirm their experience of mathematics as an esoteric and meaningless exercise in technique.

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Basic linear partial differential equations, by François Treves, Academic Press, New York, 1975, xvii + 470 pp., \$29.50.

How, and why, would one write 470 pages on "basic" linear PDE, a subject which advanced calculus texts purport to treat in 50 or 60 pages? It is not because Treves has enlarged the stock of basic equations: the standard problems and their immediate generalizations essentially fill the book. It is not because of space spent on preliminaries: distribution theory and basic functional analysis are assumed. The answer may be found by considering another question: How does one approach a typical basic problem in a modern way?

Consider a simple "mixed initial-boundary value problem" for the heat equation. The object is, given a function $u_0(x)$, $x \in [-1, 1]$, to find a function u defined on $[-1, 1] \times [0, \infty)$ such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x, 0) = u_0(x), \qquad u(\pm 1, t) \equiv 0. \tag{1}$$

Let us look at (1) as an ordinary differential equation for a vector-valued function. We denote by A the linear operator $(d/dx)^2$, with domain a suitable space of functions on $[-1, 1]$ which vanish at the endpoints. We let X be a space of functions containing the domain of A and the initial value u_0, and look for u: $[0, \infty] \to X$ such that

$$\frac{du}{dt} = Au, \qquad u(0) = u_0. \tag{2}$$

FIRST SOLUTION. Find the eigenvalues $\{\lambda_j\}$ and eigenfunctions $\{\phi_j\}$ of A, and expand $u(t) \sim \sum a_j(t)\phi_j$. This leads to a (uniquely solvable) collection of ODE's for the numerical coefficients a_j.

SECOND SOLUTION. Argue formally that u ought to be of the form $u(t) = e^{tA}u_0$, where the exponential ought to be given by the Cauchy integral formula

$$U(t) = e^{tA} = (2\pi i)^{-1}\int_\Gamma e^t(\lambda I - A)^{-1}\,d\lambda. \tag{3}$$

Since A is a negative selfadjoint operator, the resolvent $(\lambda I - A)^{-1}$ satisfies conditions guaranteeing that $\{U(t)\}$ is a holomorphic semigroup.

THIRD SOLUTION. Argue formally that (2) implies that the Laplace transform $\tilde{u}$ of the solution u satisfies

$$\lambda\tilde{u}(\lambda) = u_0 + A\tilde{u}(\lambda),$$

so that the inversion formula gives

$$u(t) = (2\pi i)^{-1}\int_\Gamma e^{t\lambda}(\lambda I - A)^{-1}u_0\,d\lambda. \tag{3$'$}$$

FOURTH SOLUTION. Construct a family A_h of finite difference approximations to A. Each A_h acts in a finite dimensional space, so $u_h' = A_h u_h$ is a system of ODE's. The solutions u_h' should converge in some sense to a solution u.

FIFTH SOLUTION. Choose a sequence of projections P_n projecting X onto finite dimensional subspaces X_n whose union is dense in X. Then $u_n' = P_n A u_n$ is a system of ODE's having a solution u_n with values in X_n. Again the u_n should converge to a solution u.

SIXTH SOLUTION. Let V_0 denote the space of smooth functions on $[-1, 1] \times [0, \infty)$ which have compact support and which vanish for $x = \pm 1$, and let V be the completion of V_0 with respect to the norm defined by the inner product

$$\langle v, w\rangle = \iint \frac{\partial v}{\partial x}\frac{\partial \bar{w}}{\partial x} + \int_{-1}^{1} v(x,0)\bar{w}(x,0)\,dx.$$

Given $v \in V_0$, there is a unique $Tv \in V$ such that

$$\langle w, Tv\rangle = \iint\left(\frac{\partial w}{\partial x}\frac{\partial \bar{v}}{\partial x} - w\frac{\partial v}{\partial t}\right), \text{ all } w \in V.$$

Then (2) may be interpreted as

$$\int_{-1}^{1} u_0(x)\bar{v}(x,0)\,dx = \langle u, Tv\rangle, \qquad \text{all } v \in V_0. \tag{4}$$

But T has a bounded inverse in V and the left side of (4) defines a bounded functional on V, so there is a unique $u \in V$ satisfying (4). In some sense, u is our desired solution.

A solution is a solution–does it matter how we find it? Even for our particular problem (or the inhomogeneous version $u' = Au + f$), results obtained most readily from different methods may be difficult to compare; for example, the last method requires less smoothness of the data and yields less smoothness for the solution than the second. Furthermore, the generalizations of the methods differ greatly in scope. The first three methods (eigenfunction expansion, semigroup, Laplace transform) are in no way restricted to differential operators A, but do not extend readily to operators which vary with t. The last three (finite difference, Galerkin, variational) use more specific structure of A, but not its independence of t.

The moral is that in PDE, methods may be as important as theorems. Faced with so many choices of method, the author of textbook or monograph is tempted to present only those most appealing to himself. With skill, one may even make the chosen methods appear to be the unique natural and proper ones.

What might one wish instead from a text for graduate students or mathematicians with some knowledge of modern analysis and a desire to understand something of present PDE? It should introduce a variety of methods in current use, applying them to classical problems and giving some indication of their more general scope; numerous exercises should illustrate the development and explore other avenues; ample references to further developments and to sources in monographs and articles should guide further study. Of course one would also prefer that the author write clearly and patiently, that he provide motivation, and that he be willing to pursue many threads at the expense of leaving loose ends.

Treves meets these requirements admirably, with the regrettable exception of adequate references. Essentially all the classical results for the wave, heat, and Laplace operators are obtained in modern language and by modern methods. The (partial) Fourier transform for tempered distributions is used repeatedly. Necessary and sufficient conditions for the Cauchy problem for a constant coefficient first-order system to be well-posed are stated and almost proved. The Cauchy-Kowalewsky theorem is proved by the slick method of Treves, Ovsjannikov, et al. The Dirichlet problem is treated by the variational method and the classical results recovered via Stampacchia's weak maximum principle. There are discussions of random walks and Brownian motion, of spherical harmonics, and of general elliptic boundary value problems. All the methods discussed above are invoked in a variety of contexts.

There are arguable points and missed opportunities. Treves does not show how the heat and Schrödinger fundamental solutions are related by analytic continuation, a relation in vogue in physical theory now and one which would have allowed evaluation of the constant in the Schrödinger fundamental solution without handwaving. The passage from variational to classical results for the Dirichlet problem seems an interesting but unnatural *tour de force*. The existence of a fundamental solution for any constant coefficient operator is mentioned, but with no reference or name. Hörmander's characterization of constant coefficient hypoelliptic equations is not mentioned. There is no discussion of, or references for, questions of hypoellipticity or local solvability of general equations, noncoercive boundary value problems, nonlinear versions of any results or methods, pseudodifferential or Fourier integral operators. While it would be unreasonable to expect more than a very brief discussion or passing reference for most of these omissions, it is unfortunate not to have that much.

The lament of the previous paragraph is that a very good text is not still better. In his preface, Treves cites two aims: "recalling the classical material to the modern analyst, in a language he can understand," and "exploiting the [classical] material, with the wealth of examples it provides, as an introduction to the modern theories." Anyone sympathetic to these aims would do well to read the entire preface, and the book.

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Simple Noetherian Rings, by John Cozzens and Carl Faith, Cambridge Tracts in Mathematics, no. 69, Cambridge Univ. Press, New York and London, 1975, xvii + 135 pp., \$12.95.

This book is concerned with a class of rings which are "simple" only in a standard technical sense. Speaking descriptively, it would be much more appropriate to entitle this material *Complicated Noetherian Rings*. Technically, a simple ring is a nonzero ring R (associative with unit, as far as this book is concerned) in which the only two-sided ideals are the two trivial ones, 0 and R. (When dealing with rings without unit, one assumes in addition that R does not have zero multiplication, i.e., $R^2 \neq 0$.) The most basic class of simple rings consists of the division rings. Although the structure of division rings is already enormously complex, one considers the division rings to be "known" in the context of general rings, and tries to relate the structure of larger classes of rings to the class of division rings in various ways. In order to be able to say much at all about simple rings in general, some chain condition is usually imposed, such as the artinian condition (any descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of one-sided ideals is ultimately constant, i.e., $I_n = I_{n+1} = \cdots$ for some n) or the noetherian condition (any ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of one-sided ideals is ultimately constant).

The first (and most widely used) general structure theorem for simple rings is of course the Wedderburn-Artin Theorem: *Any simple artinian ring R is isomorphic to the ring of all $n \times n$ matrices over some division ring D, and both n and D are uniquely determined by R.* Alternatively stated, this theorem says that R is isomorphic to the endomorphism ring of a finite-dimensional vector space over D. Because of the Hopkins-Levitzki Theorem, which states that every artinian ring is also noetherian, the Wedderburn-Artin Theorem characterizes a portion of the class of simple noetherian rings. That not all simple