AUTOMORPHISMS OF SEMIGROUPS
OF COMPLEXES OF ABELIAN GROUPS

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In the study of any algebraic system, one of the first objects investigated is
its automorphism group [1]. The automorphisms of groups have been investigat-
ed so extensively that we make no attempt at a list of references. In [4], A. R.
Richardson studied the automorphisms of groupoids. Semigroups of finite
complexes in groups are central to the study of retractable groups [2], and here
we announce some properties of the automorphism group of such a semigroup in
the case in which the underlying group is abelian. In particular, when the underly-
ing group is cyclic we classify these automorphism groups.

If \( G \) is a group, then the collection \( F(G) \) of all finite nonempty subsets of
\( G \) is a semigroup, where \( AB = \{ab | a \in A \text{ and } b \in B \} \). Each automorphism \( \alpha \)
of \( G \) induces an automorphism \( \alpha^* \) of \( F(G) \) where \( A\alpha^* = \{a\alpha | a \in A \} \). An auto-
morphism of \( F(G) \) of this type will be called a standard automorphism of \( F(G) \).

**Theorem 1.** If \( \varphi \) is an automorphism of \( F(G) \), then \( \varphi \) is a standard auto-
morphism if and only if \( \varphi \) is inclusion preserving.

A homomorphism \( \sigma \) of \( F(G) \) into \( G \) such that \( \{g\} \sigma = g \) for every \( g \) in \( G \)
is called a retraction of \( G \). A group \( G \) is called retractable if it admits a retraction.
The concept of a retractable group was introduced in [2] where it was shown
that the class of retractable groups is a proper subclass of the class of torsion free
groups and the class of lattice-ordered groups is a proper subclass of the class of
retractable groups. Hence, the class of torsion free abelian groups is a proper
subclass of the class of retractable groups. “It seems to be a rather difficult prob-
lem to determine all abelian groups with commutative endomorphism ring”
[3, p. 205]. If \( G \) is a torsion free abelian group then it is easy to show that the
automorphism group of \( F(G) \) is nonabelian.

**Theorem 2.** If \( G \) is an abelian group, \( \sigma \) is a retraction of \( G \), and \( \varphi_\sigma \) is
given by

\[
A \varphi_\sigma = (A\sigma)A(A^{-1}\sigma)
\]

for every \( A \in F(G) \), then \( \varphi_\sigma \) is an automorphism of \( F(G) \). Moreover,

(i) if \( \varphi_\sigma \) is not the identity automorphism, then \( \varphi_\sigma \) is a nonstandard
automorphism of \( F(G) \) of infinite order;

(ii) the automorphism group of $F(G)$ is an infinite nonabelian group.

Let $Z$ denote the additive group of integers. In [2] the collection of retractions of $Z$ was completely determined. If $k \in Z$ and if $\sigma_k$ is defined by $A\sigma_k = (k + 1)\max A - k \min A$ for all $A \in F(Z)$, then $\{\sigma_k | k \in Z\}$ is the collection of retractions of $Z$. We have shown that the automorphisms of $F(Z)$ that are induced by retractions of $Z$ are contained in the cyclic subgroup generated by $\sigma_{-1}$. With this information we are able to show

**Theorem 3.** The automorphism group of $F(Z)$ is isomorphic to a nonabelian splitting extension of the integers by the Klein four-group.

If $n$ is a natural number, let $Z_n$ denote the group of integers modulo $n$. Clearly, the automorphism groups of $F(Z_1)$ and $F(Z_2)$ are trivial. It can be shown that $F(Z_3), F(Z_4),$ and $F(Z_5)$ admit nonstandard automorphisms.

**Theorem 4.** If $n$ is a natural number, $n \geq 6$, then $F(Z_n)$ admits only standard automorphisms and hence the automorphism group of $F(Z_n)$ is isomorphic to the group of automorphisms of $Z_n$.

The proofs of the above results, as well as others, are computational and will appear elsewhere.

**BIBLIOGRAPHY**


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