DISCRETE SPECTRUM OF THE WEIL REPRESENTATION

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Communicated by J. A. Wolf, September 10, 1976

1. Weil representation. Let $Q$ be a nondegenerate quadratic form on $\mathbb{R}^k$. Let $O(Q)$ be the orthogonal group of $Q$. One owes to A. Weil [4] the construction of a certain unitary representation $\pi_Q$ of the group $\overline{SL}_2 \times O(Q)$ in $L^2(\mathbb{R}^k)$, where $\overline{SL}_2$ is a two fold covering of $SL_2(\mathbb{R})$, i.e. given by pairs $(g, e)$ with $g \in SL_2(\mathbb{R})$ and $e = \pm 1$ satisfying the group law $(g, e)(g', e') = (gg', V(g, g') ee')$, where $V$ is the Kubota cocycle on $SL_2(\mathbb{R})$ (with values in $\mathbb{Z}_2$). Let $w_0 \in \overline{SL}_2$ be the element $([0 1 \ -1 0], -1)$. Then $\pi_Q$ is given by

\begin{equation}
\pi_Q(w_0) \varphi(X) = \delta_Q \hat{\varphi}(-M_Q(X)), \varphi \in L^2(\mathbb{R}^k),
\end{equation}

where $M_Q \in \text{Aut}(\mathbb{R}^k)$ so that $[X, M_Q(Y)] = Q(X, Y)$ for all $X, Y \in \mathbb{R}^k$ (with $[,]$ the usual dot product on $\mathbb{R}^k$) and $\delta_Q = |\det Q|^{-1/2} u_Q$ with $u_Q$ a certain eighth root of unity determined explicitly in [2]. Moreover, $\wedge$ denotes the Fourier transform on $L^2(\mathbb{R}^k)$. Also we have

\begin{equation}
\pi_Q\left(\begin{bmatrix}
\alpha & \beta \\
0 & \alpha^{-1}
\end{bmatrix}, 1\right) \varphi(X) = |\alpha|^{k/2} e^{\sqrt{-1} \pi \beta \alpha Q(X, X)} \varphi(\alpha X), \quad \text{with } \alpha > 0
\end{equation}

and

\begin{equation}
\pi_Q(g) \varphi(X) = \varphi(g^{-1}X) \quad \text{for } g \in O(Q).
\end{equation}

Then (i), (ii), and (iii) determine $\pi_Q$ explicitly. The main problem is to give a spectral decomposition of $\pi_Q$.

2. Discrete spectrum of $\pi_Q$. Let $\tilde{K}$ be the maximal compact subgroup of $\overline{SL}_2$ given by

\begin{equation}
\left\{ \left( \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}, e \right) \mid -\pi \leq \theta < \pi, e = \pm 1 \right\}.
\end{equation}

Then every unitary character of $K$ is given by

$$k(\theta, e) \sim (\text{sgn } e)^2 m e^{\sqrt{-1} m \theta} \quad \text{with } m \in \frac{1}{2}\mathbb{Z}.$$ 

We let

$$A = \left\{ a(r) = \left( \begin{bmatrix}
r & 0 \\
0 & r^{-1}
\end{bmatrix}, 1 \right) \mid r > 0 \right\}$$
and

\[ N = \begin{cases} n(x) = \left[ \begin{array}{c} x \\ 1 \\ 0 \end{array} \right], & x \in \mathbb{R} \end{cases} \]

Let \( a, n, \) and \( f \) be the infinitesimal generators of \( A, N, \) and \( K, \) respectively. Then

\[ \omega_{\widetilde{S}1_2} = -f^2 + \alpha^2 + (n + \text{Ad}(\omega_0)n)^2 \]

is the Casimir element of \( \widetilde{S}1_2. \) We let

\[ E_+ = f + \sqrt{-1}(n + \text{Ad}(\omega_0)n) \quad \text{and} \quad E_- = k - \sqrt{-1}(n + \text{Ad}(\omega_0)n). \]

We assume that \( Q \) has inertia type \((a, b)\) where \( a \geq b \geq 1 \) and \( a + b = k \geq 3. \) Then we choose a splitting of \( Q \) on \( \mathbb{R}^k = \mathbb{R}^a \oplus \mathbb{R}^b \) so that \( X = X_+ + X_- \) with \( X_+ \in \mathbb{R}^a, X_- \in \mathbb{R}^b \) and \( Q(X, X) = ||X_+||^2 - ||X_-||^2 \) (\( || \cdot || \) is usual length of vector in \( \mathbb{R}^k \)).

We consider \( F_Q(X) = \{ \varphi \in \text{F} \mathcal{O} \mid \varphi \text{ is } C^\infty \text{ vectors in } L^2(\mathbb{R}^k) \} \) where \( F_G \) is the space of \( C^\infty \) vectors in \( L^2(\mathbb{R}^k) \) of \( \pi_Q. \) Let \( \Omega_+ = \{ X \mid Q(X, X) > 0 \} \) and \( \Omega_- = \{ X \mid Q(X, X) < 0 \} \).

**Theorem 1.** The spaces \( F_Q^+(X) = \{ \varphi \in F_Q \mid \omega_{\widetilde{S}1_2} \cdot \varphi = \lambda \varphi \} \) where \( F_Q \) is the space of \( C^\infty \) vectors in \( L^2(\mathbb{R}^k) \) of \( \pi_Q. \) Let \( \Omega_+ = \{ X \mid Q(X, X) > 0 \} \) and \( \Omega_- = \{ X \mid Q(X, X) < 0 \} \).

We consider \( F_Q^-(X) = \{ \varphi \in F_Q \mid \omega_{\widetilde{S}1_2} \cdot \varphi = -\lambda \varphi \} \) and \( F_Q^-(X) \) (if nonzero) determine topologically irreducible representations of \( \widetilde{S}1_2 \times O(Q) \) which are inequivalent. Also \( F_Q^-(X) \) is the direct sum of \( F_Q^+(X) \) and \( F_Q^-(X). \)

We let

\[ L^2(\text{Whit}) = \left\{ f: \widetilde{S}1_2 \rightarrow \mathbb{C} \mid f(gn(x)) = f(g)e^{2\pi \sqrt{-1}x} \right\}, \]

for all \( g \in \widetilde{S}1_2, x \in \mathbb{R} \) and \( \int_{\widetilde{S}1_2/N} |f(g)|^2 \, d\mu(g) < \infty \),

where \( d\mu \) is an \( \widetilde{S}1_2 \) invariant measure on \( \widetilde{S}1_2/N. \) We consider the subspace

\[ L^2(\text{Whit}) = \left\{ \psi \in L^2(\text{Whit}) \mid \omega_{\widetilde{S}1_2} \ast \psi = \lambda \psi \right\}. \]

("Discrete spectrum" means the sum of all those irreducible representations of \( \widetilde{S}1_2 \) which occur discretely in \( L^2(\text{Whit}) \))

**Theorem 2.** The discrete spectrum of \( L^2(\text{Whit}) \) is the direct sum

\[ \bigoplus_{s \in \mathbb{A}} L^2(\text{Whit})_{s^2}, \]

where \( \mathbb{A} = \{ \frac{s}{2}m > 0 \mid m \in \mathbb{Z} \}. \) Moreover, each \( L^2(\text{Whit})_{s^2} \) is \( \widetilde{S}1_2 \) irreducible and corresponds to a square integrable representation of \( \widetilde{S}1_2. \)
Theorem 3. The space $F_Q^+(\lambda) \neq 0$ if and only if $\lambda = s^2 - 2s$ with $s \in \widetilde{A} - \{\frac{1}{2}\}$ and $s \equiv \frac{1}{2}k \mod 1$. The representation of $\widetilde{SL}_2 \times O(Q)$ in $F_Q^+(s^2 - 2s)$ is equivalent to the tensor product of $L^2(\text{Whit})_{s^2 - 2s} \otimes \Lambda_s^+$, where $\Lambda_s^+ = \{\varphi \in F_Q | \varphi = \sqrt{-1} s \varphi$ and $E_+ \varphi = 0\}$. Moreover, $\Lambda_s^+$ is an irreducible $O(Q)$ module.

We note that for the case $k = 3$ an analogous tensor product as in Theorem 3 is discussed in [1].

Remark 1. If $b = 1$, then $F_Q^-(\lambda) = 0$ for all $\lambda$. And if $b > 1$, then as in Theorem 2, $F_Q^-(\lambda) \neq 0$ if and only if $\lambda = s^2 - 2s$ with $s \in \widetilde{A} - \{\frac{1}{2}\}$ and $s \equiv \frac{1}{2}k \mod 1$. Similarly $F_Q^-(s^2 - 2s)$ is $\widetilde{SL}_2 \times O(Q)$ equivalent to the tensor product $L^2(\text{Whit})_{s^2 - 2s} \otimes \Lambda_s^-$, with $L^2(\text{Whit})_{s^2 - 2s}^*$ the representation of $\widetilde{SL}_2$ in $L^2(\text{Whit})_{s^2 - 2s}^*$ after conjugation by the unique outer automorphism of $\widetilde{SL}_2$, and $\Lambda_s^- = \{\varphi \in F_Q | \varphi = -\sqrt{-1} s \varphi, E_- \varphi = 0\}$. Then the space $\Lambda_s^+$ is characterized in several ways.

Theorem 4. $\Lambda_s^+$ is $O(Q)$ equivalent to the representation of $O(Q)$ in the spaces $\{\beta \in L^2(\Gamma_1) \mid W^+_m \ast \beta = (s^2 - 2s + k - \frac{1}{4}k^2)\beta\}$ where $\Gamma_1$ is the hyperboloid $\{X \in \mathbb{R}^k \mid Q(X, X) = 1\}$ and $W^+_m$ the Laplace Beltrami operator on $\Gamma_1$ determined by the separation of variables of

$$\partial(Q) = \frac{\partial^2}{\partial t^2} + \frac{k - 1}{t} \frac{\partial}{\partial t} \frac{1}{t^2} \ W^+_m$$

(with $X = t \cdot \xi, \xi \in \Gamma_1$).

Remark 2. We note here results on the discrete spectrum of the hyperboloid similar to Theorem 4 are obtained in [3] in a different framework.

We let $K$ be the maximal compact subgroup of $O(Q)$. Then $K$ is isomorphic to the product $O(a) \times O(b)$, where $O(t)$ is the standard orthogonal group in $t$ variables. We consider the family of irreducible representations $[s_1]_a \otimes [s_2]_b$ of $K$, where $[x]_t$ denotes the representation of $O(t)$ on spherical harmonics of degree $t$. Then let $E_Q(s^2 - 2s, m, s_1, s_2)$ be the $K \times K$ isotypic component in $F_Q^+(s^2 - 2s)$ which transforms according to the character $k(\theta, \epsilon) \mapsto (\text{sgn} \epsilon)^2 m c^\sqrt{-1} \theta m$ on $\widetilde{K}$ and according to $[s_1]_a \otimes [s_2]_b$ on $K$.

Theorem 5. The space of $\widetilde{K} \times K$ finite vectors in $F_Q^+(s^2 - 2s)$ is the direct sum of the $E_Q(s^2 - 2s, m, s_1, s_2)$, where $m = s + 2j$, $j$ a nonnegative integer and $s_1$ and $s_2$ satisfy the relation $s_1 - s_2 = s - \frac{1}{2}(a - b) + 2j$. Moreover, each space $E_Q(s^2 - 2s, s + 2j, s_1, s_2)$ is spanned by elements of the form (determined on $\Omega_+$)
\[ \psi_{s,f}(Q(X, X))Q(X, X)^{s-1}e^{-\pi Q(X,x)}\|X_+\|^{s+2}\|X_-\|^{-(s+k/2+s_2-2)}, \]
\[ 2F_1\left( \frac{1}{2} (s + s_1 + s_2) + \frac{1}{4} k - 1, -j, s_2 + \frac{1}{2} b, \left( \frac{\|X_-\|}{\|X_+\|} \right)^2 \right) \cdot P_{s_1} \left( \frac{X_+}{\|X_+\|} \right) P_{s_2} \left( \frac{X_-}{\|X_-\|} \right), \]
\]
(2.1)

where \( 2F_1 \) is the usual hypergeometric function, \( P_{s_1} \) and \( P_{s_2} \) are harmonic polynomials of degree \( s_1 \) and degree \( s_2 \) in \( \mathbb{R}^a \) and \( \mathbb{R}^b \), respectively, and \( \psi_{s,f}(u) \) is the polynomial \( \sum_{\nu=0}^{\nu} c_{\nu} u^{d-\nu} \) with
\[ c_{\nu} = \frac{(-1)^{\nu}}{2^{\nu} \nu!} \frac{\Gamma(s+j)}{\Gamma(s+j-\nu)} \frac{j!}{(j-\nu)!}. \]

As an important consequence of Theorem 5 we deduce growth and continuity properties of \( \widetilde{K} \times K \) finite vectors in \( F_{\mathbb{Q}}^+(s^2 - 2s) \).

**Corollary to Theorem 5.** Every \( \widetilde{K} \times K \) finite function \( \varphi \) in \( F_{\mathbb{Q}}^+(s^2 - 2s) \) extends uniquely to a continuous function on \( \mathbb{R}^k \setminus \{0\} \) which vanishes identically on \( (\Omega_\setminus \cup \Gamma_0) \setminus \{0\} \). Moreover, if \( s > \frac{k}{2} \), then \( \varphi \) extends uniquely to a continuous function on \( \mathbb{R}^k \) which vanishes identically on \( \Omega_\setminus \cup \Gamma_0 \). Also such a \( \varphi \) satisfies the Poisson Summation Formula Property, that is, for any lattice \( L \subset \mathbb{R}^k \) with \( Q(L, L) \subset \mathbb{Z} \), the integers,
\[ F(X) = \sum_{\xi \in L} \varphi(X + \xi), \]
(2.2)
is continuous (with the summation satisfying absolute convergence) on \( \mathbb{R}^k \) and \( \sum_{\xi \in L} \hat{\varphi}(\xi^*) \) is absolutely convergent (\( L^* \) dual lattice to \( L \)).

We remark that similar types of statements hold for \( \widetilde{K} \times K \) functions \( f \in F_{\mathbb{Q}}^+(s^2 - 2s) \).

**BIBLIOGRAPHY**


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