TOWARDS ALGEBRAIC COBORDISM

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Abstract. A new description of cobordism is given and, by analogy, cobordism theories are defined for an arbitrary ring.

1. Let $A$ be a ring with a unit. A cohomology theory, $MA$, might reasonably be called "the algebraic cobordism of $A"$ if

(i) geometry over $A$ gave rise to elements in $\pi_*(MA)$, and

(ii) the existence of Chern classes for $A$ induced a transformation of cohomology theories from $MA$ to the algebraic $K$-theory of $A$.

Below I sketch the construction of theories which often satisfy (i) and (ii). Details will appear in [2], [3].

Let $X$ be a homotopy associative and commutative $\Omega$-space. Let $T \subset H\pi^*(X)$ be a finite subset of homogeneous elements. To this data is associated a periodic, commutative ring spectrum $X(T)$. $X(T)^*$ is the associated cohomology theory. For example, when $X = BU$ and $T$ consists of the generator $B \in \pi_2(BU)$, then $X(T)_{2k} = \Sigma^2 BU$ and $\epsilon_{2k} : \Sigma^2 X(T)_{2k} \to X(T)_{2k+2}$ is equal to

$$
\Sigma^2(\Sigma^2 BU) \xrightarrow{h} \Sigma^2(S^2 \times BU) \xrightarrow{\Sigma^2(B \oplus \text{id})} \Sigma^2(BU).
$$

Here $h$ is a Hopf construction and "id" is the identity map of $BU$.

When $X = BGLA^+$ for a ring $A$ and $T \subset H\pi^*(BGLA^+)$, $X(T)^*$ is called the algebraic cobordism of $A$ associated with $T$. The terminology is motivated by (a)—(c) of the following result:

**Theorem 1.1.** Suppose $\dim Y < \infty$; then:

(a) $BU(T)^0(Y) \simeq MU^2(Y)$ if $T = \langle \text{generator of } \pi_2(BU) \rangle$;

(b) $BSp(T)^0(Y) \simeq MSp^4(Y)$ if $T = \langle \text{generator of } \pi_4(BSp) \rangle$;

(c) $BO(T)^0(Y) \simeq MO^*(Y)$ if $T = \langle \text{generator of } \pi_1(BO) \rangle$;

(d) if $F$ is a finite field and $T$ is a subset of $K_*(F)$ then $BGLF^+(T)^0(Y) \simeq 0$;

(e) if $T = \langle \text{generator of } K_1(Z) \rangle$ then $BGLZ^+(T)^0(Y)$ in general is a non-trivial group in which each element is of order 2.


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Theorem 1.1 relates $K$-theory and cobordism very satisfactorily. For example, Adams operations in $KU^*$ induce Adams operations in $MU^*$ while Adams idempotents in $KU^*$ induce Adams idempotents in $MU^*$.

The starting point for Theorem 1.1 is the following:

**Theorem 1.2.** If $1 \leq n \leq \infty$ there exist stable equivalences

(i) $BU(n) = \bigvee_{1 = k}^n MU(k)$,

(ii) $BSp(n) = \bigvee_{1 = k}^n MSp(k)$,

(iii) $BO(2n) = \bigvee_{1 = k}^n BO(2k)/BO(2k - 2)$ and

(iv) $BSO(2n + 1) = \bigvee_{1 = k}^n BSO(2k + 1)/BSO(2k - 1)$ when localised away from 2.

2.1. **Sketch of Proof of Theorem 1.2.** The Becker-Gottlieb transfer is used to embed each classifying space, as a filtered space, into $QW = \lim\Omega^n \Sigma^n W$ for suitable $W$. For example $BU$ is embedded in $QBU(1)$. The decompositions then follow from the decomposition theorem of [1].

2.2. **Sketch of Theorem 1.1.** Consider the unitary example. Then

$$BU(T)^0(Y) = \lim_{N} \{\Sigma^{2N} Y, BU\}$$

where $\{ , \}$ means homotopy classes of $S$-maps. Hence, by Theorem 1.2, if $\dim Y \leq 4t$

$$(2.3) \quad BU(T)^0(Y) \simeq \lim_{M} \bigoplus_{t + M < k} \{\Sigma^{M} Y, MU(k)\} \oplus \prod_{t-M \leq l} MU^{2l}(Y).$$

A careful study of the $S$-equivalences of Theorem 1.2 and some obstruction theory shows that only the cobordism part of (2.3) remains in the limit.

**REFERENCES**


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