A CHARACTERIZATION OF HARMONIC IMMERSIONS
OF SURFACES

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Let $S$ be an oriented surface with Riemannian metric $ds^2$, and $M^n$ a Rie-
mannian manifold of dimension $n \geq 2$. We present here a characterization of
harmonic immersions $f: S \to M^n$ which sheds some light on their differential
geometric properties. While $C^\infty$ smoothness is assumed throughout, less is
needed.

To work on the Riemann surface determined by $ds^2$ on $S$, use conformal
parameters $z = x_1 + ix_2$ which correspond to $ds^2$-isothermal coordinates $x_1, x_2$
on $S$. Given any local coordinates on $M^n$, write $f = (f^\alpha)$ and $f_i^\alpha = \partial f^\alpha / \partial x_i$
where $i = 1, 2$ and $\alpha, \beta, \gamma = 1, 2, \ldots, n$. An immersion $f: S \to M^n$ is har-
monic if and only if for each $\alpha$ and for any $ds^2$-isothermal coordinates $x_1, x_2$
on $S$
$$\partial^2 f^\alpha / \partial x_i^2 + \Gamma^\alpha_{\beta\gamma} f_i^\beta f_i^\gamma = 0,$$
where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for the metric on $M^n$, and one sums on
the indices $\beta, \gamma$ and $i$.

To any real quadratic form $X = l_{ij} dx_i dx_j$ on $S$, associate on $R$ the qua-
dratic differential $\Omega(X, R)$ and the conformal metric $\Gamma(X, R)$ given by $4\Omega(X, R)$
$= (l_{11} - l_{22} - 2il_{12})dz^2$ and $2\Gamma(X, R) = (l_{11} + l_{22})|dz|^2$ respectively. Thus
$X = 2 \text{Re } \Omega + \Gamma$ on $R$. (See [10].) Call $\Omega(X, R)$ holomorphic if and only if
the coefficient of $dz^2$ is complex analytic in $z$ for every conformal parameter $z$
on $R$. An immersion $f: S \to M^n$ yields many quadratic forms of interest,
among them the induced metric $I$, and the second fundamental forms $II(N)$ de-
termined by choices of a unit normal vector field $N$.

**DEFINITION.** An immersion $f: S \to M^n$ is $R$-minimal if and only if
$\Omega(I, R)$ is holomorphic, and $\Gamma(II(N), R) \equiv 0$ for any choice (local or global) of
a unit normal vector field $N$.

An $R$-minimal immersion is minimal if and only if $R$ is the Riemann sur-
face $R_1$ determined on $S$ by $I$. It is known that a conformal immersion $f: S \to
M^n$ is harmonic if and only if it is minimal. Indeed, this is established in [2]
independent of the dimensions of $S$ and $M^n$. By analogy, we have the following


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THEOREM. An immersion \( f: S \to M^n \) is harmonic if and only if it is \( R \)-minimal.

This result is known for maps \( f: S \to M^2 \). (See [4] for references.) It is also known that \( \Omega(I, R) \) must be holomorphic for any harmonic map \( f: S \to M^n \), so that the only harmonic maps of the 2-sphere must be minimal ([2] and [8]).

We consider immersions here to provide (when \( n \geq 3 \)) a well-defined \((n - 2)\)-dimensional normal space everywhere.

Note that \( \Gamma(\Pi(N), R) \equiv 0 \) for all \( N \) means that the trace of \( \Pi(N) \) with respect to \( ds^2 \) vanishes for all \( N \). When \( ds^2 \neq I \), this condition alone forces a minimal immersion, for it says that the mean curvature vector [11, p. 13] vanishes. Indeed, by our Theorem, the "mean curvature vector" formed with \( ds^2 \) in place of \( I \) vanishes for any harmonic immersion \( f: S \to M^n \). The converse can fail when \( R \neq R_1 \). For example, if \( S \) is immersed in \( E^3 \) with Gauss curvature \( K \equiv -1 \), the usual asymptotic Tchebychev coordinates [9, p. 528] are \( \Pi' \)-isothermal, where \( \sqrt{H^2 + 1} \Pi' = H \Pi + I \), with \( H \) mean curvature. Here \( \Gamma(II, R_{II}) \equiv 0 \) but \( \Omega(I, R_{II}) \) is not holomorphic. Similarly, \( \Omega(I, R) \) holomorphic does not imply \( \Gamma(II(N), R) \equiv 0 \) for any \( N \). This is obvious when \( R = R_1 \). Less trivially, if \( S \) is immersed in \( E^3 \) with \( K \equiv 1 \), then \( \Omega(I, R_{II}) \neq 0 \) is holomorphic, but \( \Gamma(II, R_{II}) \equiv II \) does not vanish [5].

The proof of the theorem is elementary, using the Gauss equations [5, p. 160]. Some results which follow from the theorem are stated below for the special case \( n = 3 \). Full details and proofs will appear elsewhere. Hereafter, \( f: S \to M^3 \) is an immersion with fundamental forms \( I \) and \( II \), mean curvature \( H \), Gauss curvature \( K \) and intrinsic curvature \( K(I) \). Denote by \( K \) the sectional curvature of \( M^3 \) for planes tangent to \( S \), by \( \Lambda = gI + hII \) any positive definite linear combination with real valued coefficients \( g \) and \( h \), by \( R \) the Riemann surface determined on \( S \) by \( ds^2 \) and by \( R \) an arbitrary Riemann surface on \( S \). The form \( II' \) given by \( \sqrt{H^2 - K} II' = HII - Ki \) is positive definite wherever \( K < 0 \) [10].

Lemmas 1 and 2 reflect the separate effects of the conditions \( \Omega(I, R) \) holomorphic and \( \Gamma(II, R) \equiv 0 \). Theorem 2 includes a correction of the Corollary to Theorem 2 in [7].

**Lemma 1.** If \( \Omega = \Omega(I, R) \neq 0 \) is holomorphic, then except at isolated points where \( \Omega = 0 \), there exists a canonically determined function \( F > 0 \) on \( S \) which is \( R \)-superharmonic where \( K(I) \geq 0 \) and \( R \)-subharmonic where \( K(I) \leq 0 \) [1, p. 135].

**Lemma 2.** If \( \Gamma(II, R) \equiv 0 \) for any one \( R \) on \( S \), then \( K \leq 0 \) (so that \( K(I) \leq K \)), and \( H = 0 \) wherever \( K = 0 \).

**Theorem 1.** If \( f: S \to M^3 \) is harmonic with \( ds^2 = \Lambda \), then either \( \Lambda \propto I \), or else (except at isolated points where \( \Lambda \propto I \)) \( \Lambda \propto II' \).
THEOREM 2. If \( f: S \rightarrow M^3 \) is harmonic with \( ds^2 = II' \), \( H \) never zero and \( 0 \neq K(I) \leq 0 \), then \( H'/H \) is not bounded.

THEOREM 3. If \( f: S \rightarrow M^3 \) is harmonic with \( ds^2 = II' \) complete, \( |K/H| \) bounded and \( K(II') \leq 0 \) then \( K(II') \equiv 0 \).

THEOREM 4. If \( f: S \rightarrow M^3 \) is harmonic with \( R \) parabolic [1, p. 209], \( I \) nowhere proportional to \( ds^2 \) and \( K(I) \geq 0 \), then \( K(I) \equiv 0 \).

REFERENCES


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