THE GENERALIZED GAMMA FUNCTION, NEW HARDY SPACES, AND REPRESENTATIONS OF HOLOMORPHIC TYPE FOR THE CONFORMAL GROUP

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Operator valued generalizations of the integral formula for the classical gamma function arise in connection with noncompact semisimple, or reductive, Lie groups for which the symmetric space $G/K$ is Hermitian, and they relate to various problems in analysis, group representations, and number theory. In particular, when the holomorphic discrete series for $G$, constructed originally by Harish-Chandra [3], is realized in terms of the unbounded form of $G/K$ as a Siegel domain, the gamma function plays a decisive role (cf., [1], [2a], [2b], [2d], [6a], [6b]). Moreover, the holomorphic discrete series has an analytic continuation [7], the full extent of which is controlled by the analytic continuation of a normalized version of the gamma function. In general, however, it is only when the gamma function is scalar valued, an occurrence which accounts for but a small part of the holomorphic discrete series, that the full analytic continuation has been determined. In that specialized context, it is known from [6b] that Hardy type Hilbert spaces associated to the various boundary components of $G/K$ appear at the "integer points" in the analytic continuation.

This note announces rather complete solutions to these problems for the conformal group $G = U(2, 2)$. Specifically, we give the entire analytic continuation of the gamma function, the full extent of analytic continuation of the holomorphic discrete series, and we introduce some new vector-valued Hardy spaces.

I. The generalized gamma function. Let $A = A \times A$ where $A = GL(2, C)$ and fix a complete set of irreducible holomorphic finite-dimensional representations $\lambda$ of $A$ such that $\lambda(a_1, a_2)^* = \lambda(a_1^*, a_2^*)$. Let $\lambda$ be parametrized by a pair of highest weights $(\alpha_j + 2l_j, \sigma_j), (j = 1, 2)$, where $\sigma_j$ and $2l_j$ are integers and $l_j \geq 0$. Then $\lambda = \lambda(\cdot ; \sigma_1, \sigma_2, \lambda^0)$ where

$$\lambda(a_1, a_2) = \Delta(a_1)^{\sigma_1} \Delta(a_2)^{\sigma_2} \lambda^0(a_1, a_2)$$

with $\Delta = \det$ and $\lambda^0 = \lambda^0(\cdot ; l_1, l_2)$ a polynomial representation. Let $P$ be


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the cone of positive matrices in $\mathcal{A}$. The generalized gamma function for $G$ of weight $\lambda^0$ is the holomorphic operator-valued function defined by the absolutely convergent integral

$$\Gamma_{\lambda^0}(\alpha) = \int_{\mathcal{P}} e^{-\text{tr}(r)}\lambda^0(\mathcal{P}, r)\Delta(r)^{\alpha-2} \, dr$$

for $\text{Re}(\alpha) > 1$ and elsewhere by analytic continuation.

The following is the main technical theorem. It is proven from the Clebsch-Gordon formula by an involved calculation.

**Theorem 1.** Fix $\lambda^0$. The values $\Gamma_{\lambda^0}(\alpha)$ for $\text{Re}(\alpha) > 1$ form a commutative family of normal operators having distinct eigenvalues.

$$\gamma_{\lambda^0}^{(l)}(\alpha) = \frac{\Gamma(\alpha + 2l_1)\Gamma(\alpha + 2l_2)\Gamma(\alpha + 2l_1 + 2l_2 + 1)\Gamma(\alpha - 1)}{\Gamma(\alpha + l_1 + l_2 - l)\Gamma(\alpha + l_1 + l_2 + l + 1)}$$

indexed by $l = |l_1 - l_2|, \ldots, l_1 + l_2$.

**Corollary 1.** The function $\alpha \rightarrow \Gamma_{\lambda^0}(\alpha)^{-1}$ extends from $\text{Re}(\alpha) > 1$ to an entire function of $\alpha$.

We remark that in the special case $l_1 = 0$ or $l_2 = 0$ (cf., [2a]), $\Gamma_{\lambda^0}$ is scalar-valued.

Let $\gamma_{\lambda^0}(\alpha) = \text{tr}(\Gamma_{\lambda^0}(\alpha - 2)^{-1}\Gamma_{\lambda^0}(\alpha))$, and define the normalized gamma function $N_{\lambda^0}$ by $N_{\lambda^0}(\alpha) = \gamma_{\lambda^0}(\alpha)(\Gamma_{\lambda^0}(\alpha - 2)$.

**Corollary 2.** (i) The function $\alpha \rightarrow N_{\lambda^0}(\alpha)^{-1}$ is entire.

(ii) If either $l_1 = 0$ or $l_2 = 0$ (so $N_{\lambda^0}(\alpha)$ is scalar), then $N_{\lambda^0}(\alpha)^{-1}$ is positive for $\alpha > 1$, and $N_{\lambda^0}(1)^{-1} = 0$.

(iii) If both $l_1 \neq 0$ and $l_2 \neq 0$, then $N_{\lambda^0}(\alpha)^{-1}$ is a positive operator for $\alpha > 2$, and $N_{\lambda^0}(2)^{-1}$ is nonnegative (one eigenvalue is positive and all others vanish).

Let $P_1 = \{ r \in C^{2 \times 2} : r = r^* \geq 0 \text{ and } \Delta(r) = 0 \}$, the rank one boundary component of $P$, and set

$$\widetilde{\Gamma}(\lambda^0) = \int_{P_1} e^{-\text{tr}(r)}\lambda^0(\mathcal{P}, r) \, dm(r)$$

where $m$ is relatively $A$-invariant measure on $P_1$.

**Corollary 3.** The operator $\widetilde{\Gamma}(\lambda^0)$ is positive. In fact, $\widetilde{\Gamma}(\lambda^0) = \lim_{\alpha \rightarrow 1} (\Gamma_{\lambda^0}(\alpha) / \Gamma(\alpha - 1)) = c(\lambda^0)N_{\lambda^0}(3)$, where $c(\lambda^0)$ is real and positive.

**II. Analytic continuation of the holomorphic discrete series.** We realize $G/K$ as the Siegel upper half plane $H = S + iP$ in $C^{2 \times 2}$ where $S = \{ x \in C^{2 \times 2} : x = x^* \}$. The relative holomorphic discrete series for the universal covering group $\tilde{G}$ of $G$ consists of representations $\mathcal{T}(\cdot, \lambda)$ indexed by $\lambda$ as in (1) with $\sigma_1, \sigma_2$ real and $\alpha = \sigma_1 + \sigma_2 > 3$. $\mathcal{T}(\cdot, \lambda)$ acts in the space $\mathcal{H}(\lambda^0, \alpha)$ of
holomorphic vector-valued functions $F$ such that
\[ \|F\|_{\lambda_0, \alpha}^2 = \int_H \|\lambda^0(y, \overline{y})^{1/2}F(x + iy)\|^2 \Delta(y)^{\alpha - 4} \, dx \, dy < \infty. \]
When $\sigma_1$ and $\sigma_2$ are integers, $T(\cdot, \lambda)$ is a representation of $G$ itself. $H(\lambda^0, \alpha) \neq 0$ if and only if $\alpha > 3$. The reproducing kernel $Q_{\lambda_0, \alpha}$ for $H(\lambda^0, \alpha)$ is calculated (cf., [2d]) to be
\[ (2) \quad Q_{\lambda_0, \alpha}(z, w) = \int e^{i\text{tr}(z-w^*)}\lambda^0(r, \overline{r})^{1/2}N_{\lambda_0}(\alpha)^{-1} \lambda^0(r, \overline{r})^{1/2} \Delta(r)^{\alpha - 2} \, dr \]
for $z, w \in H$, which can be evaluated as
\[ (3) \quad Q_{\lambda_0, \alpha}(z, w) = \Delta(-i(z - w^*))^{-\alpha}(-i(z - w^*), -i(z - w^*))^{-1}. \]
Following [6b], we define the Wallach set for $G$ to consist of all $(\lambda^0, \alpha)$ such that $Q_{\lambda_0, \alpha}$ given by (3) is positive-definite.

**Theorem 2.** Fix any $\lambda^0$. Then: (i) If $l_1 = 0$ or $l_2 = 0$ (so $N_{\lambda_0}(\alpha)$ is scalar), then $(\lambda^0, \alpha)$ is in the Wallach set for $\alpha > 1$. (ii) If $l_1 \neq 0$ and $l_2 \neq 0$, then $(\lambda^0, \alpha)$ is in the Wallach set if $\alpha > 2$.

The proof is immediate from (2) and Corollary 2. In Case (i), $N_{\lambda_0}(\alpha)^{-1}$ becomes negative for $\alpha < 1$; and in Case (ii), $N_{\lambda_0}(\alpha)^{-1}$ ceases to be a nonnegative operator for $\alpha < 2$. Therefore, **Theorem 2 gives the complete analytic continuation of the holomorphic discrete series.** The special case $\lambda^0 = 1$ (for generic $G$) is due to Rossi and Vergne [6b], and Case (i) appears in [2a] and [4]. Case (ii) with $2 < \alpha < 3$ yields new Hilbert spaces $H(\lambda^0, \alpha)$ and new irreducible representations of $\widehat{G}$. The parameter $\alpha = 3$, corresponds to a space $H(\lambda^0, 3)$ which realizes the limit of holomorphic discrete series in the sense of [5].

**III. New Hardy spaces.** Let $V(\lambda^0)$ be the space of $\lambda^0$ and denote by $\mathcal{S}^2(\lambda^0, P_1)$ the Hilbert space of all holomorphic $F: H \rightarrow V(\lambda^0)$ such that
\[ \|F\|_{\lambda^0, P_1} = \sup_{t \in \mathbb{R}} \int_{S + itP_1} \|\lambda^0(y, \overline{y})^{1/2}F(x + iy + it)\|^2 \, dx \, dm(y) < \infty. \]

**Theorem 3.** $\mathcal{S}^2(\lambda^0, P_1) = H(\lambda^0, 3)$. In particular, $\mathcal{S}^2(\lambda^0, P_1) \neq 0$.

The proof follows from Corollary 3 by a Plancherel theorem argument. For $\lambda^0 = 1$, Theorem 3 is given in [6b]. There, it is also shown that discrete points appear in the Wallach set (for $\lambda^0 = 1$) beyond the range of analytic continuation. The analytic implementation of such spaces for generic $\lambda^0$ is an interesting open problem.

**REFERENCES**


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**ERRATUM, VOLUME 82**


In Volume 82, p. 421, line 14 should read: "analyzes both the form of the group of motions, and the underlying space".