Inverse Scattering for the Klein-Gordon Equation

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In this note we would like to announce recent results concerning the so-called Inverse Scattering problem for the Klein-Gordon equation in three dimensions. Complete proofs of this work will appear in [1].

We consider the Klein-Gordon equation with a linear perturbation, that is

\[ u_{tt} - \Delta u + m^2 u + q(x)u = 0 \]

in \( \Omega = \mathbb{R}^3, -\infty < t < +\infty \). Here the subscripts denote partial derivatives, \( m > 0 \) and \( \Delta \) is the Laplacian operator. The potential \( q(x) \) is assumed to be a real valued function in \( \mathbb{R}^3 \), nonnegative and satisfying certain reasonable conditions at infinity which we will specify later. The initial Cauchy data for (1) at \( t = 0 \) will be assumed to be \( C^\infty \) with compact support. In the space of such solutions of (1) we define the (total) energy of \( u \) as

\[ \|u\|_E^2 = \frac{1}{2} \int_{\mathbb{R}^3} [|\text{grad } u|^2 + u_t^2 + m^2 u^2 + q(x)u^2] \, dx \]

where \( |\text{grad } u|^2 = \sum_{j=1}^3 u_x^2_j \). It is easy to show that \( \|u\|_E \) is constant i.e. we are dealing with a conservative equation. If we assume (for example) that \( q(x) \in L^1 \cap L^\infty(\mathbb{R}^3) \) then it is well known (see for example [3] and [4]) that given a solution \( u \) of (1) there then exists a unique pair \( u_\pm \) of solutions of (1) with \( q = 0 \) such that

\[ \|u - u_\pm\|_E \to 0 \quad \text{as } t \to \pm \infty. \]

The operator which relates \( u_- \to u_+ \) is called the scattering operator and is denoted by \( S \). One want to know what can be said about \( q(x) \) if we know the operator \( S \)? This is a problem of physical relevance (see [5], [6]). If \( q(x) \) is spherically symmetric, then there has been considerable research on this problem in the past twenty five years, mainly through the Gelfand-Levitan-Marchenko approach. In dimensions higher than one, very little is known. Here, we announce a “local” uniqueness result concerning the 3-dimensional inverse problem for (1).

**Theorem.** Let \( q_1(x) \) and \( q_2(x) \) be a nonnegative continuous functions which belong to \( L^1 \cap L^\infty(\mathbb{R}^3) \). Let \( S(q_1) \) and \( S(q_2) \) denote the scattering operators associated with \( u_{tt} - \Delta u + m^2 u + q_1 u = 0 \) and \( v_{tt} - \Delta v + m^2 v + q_2 v = 0 \), respectively. Then \( S(q_1) = S(q_2) \) if and only if \( q_1 = q_2 \).


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respectively. If \( S(q_1) = S(q_2) \), then
\[
\lim_{\epsilon \to 0^+} \frac{\epsilon \| q_1 - q_2 \|}{\alpha(\epsilon q_1, \epsilon q_2)} = 0.
\]
Therefore, if \( q_1(x) \neq q_2(x) \) for some \( x \in \mathbb{R}^3 \) and the above limit is different from zero, then \( S(q_1) \neq S(q_2) \).

Here,
\[
\| q_1 - q_2 \| = \sup_{x \in \mathbb{R}^3} \sup_{r \in \mathbb{R}} \left| \int_{-\infty}^{\infty} R(x, t-r) \ast (q_1 - q_2) u_-(x, r) \, dr \right|
\]
where \( u_- \) denotes any incoming free solution of (1) (with \( q \equiv 0 \)), \( R \) the Riemann function of (1) with \( q \equiv 0 \), and \( \ast \) denotes spatial convolution, \( \alpha(q_1, q_2) \) is given by a constant times
\[
\left( \| q_1 \|_{1/3}^{1/6} \| q_1 \|_1^{1/6} + \| q_1 \|_{1/2} \| q_1 \|_1^{1/2} \right) \left( \| q_1 \|_{1/3}^{1/6} \| q_1 \|_1^{1/6} \right) + \left( \| q_2 \|_{1/3} \| q_2 \|_1^{1/6} + \| q_2 \|_{1/2} \| q_2 \|_1^{1/2} \right) \left( \| q_2 \|_{1/3} \| q_2 \|_1^{1/3} \right)
\]

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REFERENCES


