The simplest nontrivial oriented topological surface is the 2-dimensional torus. It is well known that any compact Riemann surface is topologically equivalent to the 2-sphere with handles attached, that is, to a connected sum of 2-toruses. We can consider this decomposition as corresponding to the canonical decomposition of the (skew-symmetric) intersection form of 1-homologies on the given Riemann surface.

In the case of simply-connected algebraic surfaces the intersection form of 2-homologies plays a fundamental role because it defines completely the homotopy type of the corresponding 4-dimensional topological manifold (see [1], [2]).

Performing a $\sigma$-process on the given simply-connected algebraic surface $V$ we obtain an algebraic surface $V'$ which contains a 2-dimensional homology class with self-intersection equal $-1$ (which is an odd number). Then it is well known (see [3], [4]) that there exists a basis of $H_2(V', \mathbb{Z})$ such that the corresponding intersection matrix is diagonal. The corresponding “elementary blocks” $||+1||$ and $||-1||$ are the intersection matrices of the simplest nontrivial oriented simply-connected 4-manifolds:

- $P =$ complex projective plane with its usual orientation and $Q =$ complex projective plane with orientation opposite to the usual. From the homotopy classification theorem [1], [2], it follows that $V'$ is homotopy equivalent to a connected sum of $P$'s and $Q$'s. Of course, the “ideal situation” (analogous to the mentioned above topological decomposition of compact Riemann surfaces), which we could expect, is the existence of a homeomorphism of $V'$ to this connected sum. However, there are some nondirect indications that $V'$ is homeomorphic to a connected sum of $P$'s and $Q$'s if and only if $V'$ is a rational algebraic surface. (This conjecture was formulated in [5].) The question is still open, but assuming the conjecture we can consider as a realistic aim only the problem of estimating how “far” topologically is the given nonrational simply-connected algebraic surface from an “ideal” topological model, that is, from a connected sum of $P$'s and $Q$'s.

In [6] Wall proved the following theorem: If $M_1, M_2$ are simply-connected compact 4-manifolds, which are homotopically equivalent, then there exists


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an integer \( k \geq 0 \) such that \( M_1 \# k(S^2 \times S^2) \) is diffeomorphic to \( M_2 \# k(S^2 \times S^2) \) (# is the connected sum operation).

It follows almost immediately from this result that if \( M \) is a simply-connected compact 4-manifold, then there exists an integer \( k \geq 0 \) such that \( M \# (k + 1)P \# k \cdot Q \) is diffeomorphic to \( IP \# mQ \) for some \( l, m \geq 0 \).

After the proof of his theorem Wall writes the following [6, p. 147]: "We remark that our results is a pure existence theorem; We have obtained, even in principle, no bound whatever on the integer \( k \)".

As it was remarked in [5], the operation \( M \# P \) (resp. \( M \# Q \)) where \( M \) is an oriented 4-manifold could be considered as performing of certain blowing up of some point on \( M \). We call this blowing up \( \sigma \)-process (resp. \( \sigma \)-process). (The exact definition of \( \sigma \)-process and \( \sigma \)-process is the following: In a small enough neighborhood \( N_x \) of a point \( x \in M \) we can always take local coordinates giving \( N_x \) a complex structure. This complex structure will then have an orientation the same as that of \( M \) or opposite to that of \( M \). Performing a classical \( \sigma \)-process using the local complex coordinates of \( N_x \), we get an operation which in the first case we call "\( \sigma \)-process" and in the second case "\( \sigma \)-process".)

We say that an oriented simply-connected 4-manifold \( W \) is completely decomposable (resp. almost completely decomposable) if \( W \) (resp. \( W \# P \)) is diffeomorphic to \( IP \# mQ \) for some \( l, m \geq 0 \).

Let \( M \) be an oriented compact simply-connected 4-manifold. For \((k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}, k_1 \geq 0, k_2 \geq 0 \), let \( M(k_1, k_2) \) be a 4-manifold obtained from \( M \) by \( k_1 \) \( \sigma \)-processes and \( k_2 \sigma \)-processes. Denote by \( \mathcal{W}(M) = \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}|k_1 \geq 0, k_2 \geq 0, M(k_1, k_2) \) is completely decomposable \}. It follows from the theorem of Wall that \( \mathcal{W}(M) \neq \emptyset \).

An important geometrical problem is to define minimal elements of \( \mathcal{W}(M) \) (in any natural sense). A certain step for solving this problem could be the construction of some elements of \( \mathcal{W}(M) \) in explicit form, say in terms of the 2-dimensional Betti number and of the signature of \( M \). We can prove that such a construction is possible when \( M \) admits a complex structure. The main result is

**Theorem A.** Let \( M \) be a compact simply-connected 4-manifold which admits a complex structure. Take an orientation on \( M \) corresponding to certain complex structure on it. Let \( K(X), L(X) \) be cubic polynomial defined as follows:

\[
K(X) = \tilde{K}(9(5X + 4)) - X, \quad L(X) = \tilde{L}(9(5X + 4)),
\]

where \( \tilde{K}(t) = t(t^2 - 6t + 11)/3, \tilde{L}(t) = (t - 1)(2t^2 - 4t + 3)/3 \)

\[
(K(X) = 30375X^3 + 68850X^2 + 52004X + 13092, \quad L(X) = 60750X^3 + 141750X^2 + 110265X + 28595).
\]

Denote by \( b_+ \) (resp. \( b_- \)) the number of positive (resp. negative) squares in the inter-
section form of $M$ and let $k'_1 = K(b_+), k'_2 = \max(0, L(b_+) - b_-)$. Then the pair $(k'_1, k'_2) \in \mathcal{W}(M)$.

**Remarks about the proof of Theorem A.** From the Kodaira classification of compact complex surfaces [7] it follows that if $M$ is a simply-connected compact complex surface, then there exists a nonsingular projective-algebraic complex surface $V$ such that $V$ is diffeomorphic to $M$ and one of the following three possibilities holds:

(a) $V$ is rational;
(b) $V$ is elliptic;
(c) $V$ is of general type.

In case (a) our theorem is evident. In case (b) we can prove a much stronger result.

**Theorem B.** Any simply-connected elliptic surface $V$ is almost completely decomposable. (That is, $(1, 0) \in \mathcal{W}(V)$.)

For the case (c) we first prove the following comparison theorem for topology of projective algebraic surface of given degree $n$ and nonsingular hypersurface of degree $n$ in $\mathbb{CP}^3$:

**Theorem C.** Let $V_n$ be a projective algebraic surface of degree $n$ embedded in $\mathbb{CP}^N, N \geq 5$, such that $V_n$ is not contained in a proper projective subspace of $\mathbb{CP}^N$. Suppose that $V_n$ is nonsingular or has as singularities only rational double-points. Let $h: \tilde{V}_n \to V_n$ be a minimal desingularization of $V_n$ (that is, $\tilde{V}_n$ has no exceptional curve of first kind $s$ such that $h(s)$ is a point on $V_n$). Suppose $\pi_1(\tilde{V}_n) = 0$. Denote by $Y_n$ the diffeomorphic type of a nonsingular hypersurface of degree $n$ in $\mathbb{CP}^3$.

Then

(i) $b_+(\tilde{V}_n) < b_+(Y_n), b_-(\tilde{V}_n) < b_-(Y_n)$;

(ii) $\tilde{V}_n \# [b_+(Y_n) - b_+(\tilde{V}_n) + 1]P \# [b_-\left(Y_n\right) - b_-(\tilde{V}_n)]Q$ is diffeomorphic to $Y_n \# P$.

In [5] it was proved that $Y_n \# P$ is completely decomposable. Thus we need only some estimation of a possible minimal degree for projective embeddings of $V$ in terms of $b_+(V), b_-(V)$. We obtain such an estimation from Bombieri's results on pluricanonical embeddings of algebraic surfaces of general type [8].

**Remark to Theorem B.** Note that Theorem B together with results of [5], [9], [10] show that all big explicit classes of simply-connected algebraic surfaces considered until now have the property that their elements are almost completely decomposable 4-manifolds. That is, the "theoretical" Theorem A gives much weaker results than our "empirical" knowledge.

The interesting question is, how far we can move with such "empirical achievements" in more general classes of simply-connected algebraic surfaces.


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