field generated by the character values. A measure of this discrepancy is given by what is called the *Schur index*. There are intriguing connections between algebraic number theory and the subgroup-structure of finite groups, and more recently with division algebras over the rational field, which have a common focus on the Schur index. Isaacs, in collaboration with Goldschmidt, has contributed to this subject himself, and includes some of his own work in this area, along with a number of related results.

Isaacs' book concludes with some topics which are either more specialized, or lead outside his intended scope. In particular, an introduction is given to Brauer's theory of modular characters. In this theory, the irreducible complex characters, whose values all lie in the ring of algebraic integers R in some number field K, are classified into subsets, called p-blocks, according to their behaviour with respect to some prime ideal P in R, dividing a fixed rational prime p. This theory, which achieves a finer analysis of the characters, has been used to prove results not originally accessible by the method of ordinary character theory, such as the result of Brauer and Suzuki that the quaternion group of order 8 cannot be the 2-Sylow group of a finite simple group. Brauer's theory involves the study of representations of groups in fields of characteristic $p > 0$, where the representations are not necessarily completely reducible, and the study of indecomposable modules as well as irreducible ones, becomes of crucial importance. The point of view required for the analysis of indecomposable modules is presented more fully in [**2**], [**3**] and [**7**].

Isaacs' book includes extensive lists of carefully thoughtout problems, which extend the scope of the book considerably. The book is a pleasure to read. There is no question but that it will become, and deserves to be, a widely used textbook and reference, as well as a place where curious mathematicians from other fields can find a clear and authoritative introduction to a fascinating part of group theory.

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CHARLES W. CURTIS

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Mathematical logic, by J. Donald Monk, Springer-Verlag, New York, x + 531 pp., $19.80.

"On the banks of the Rhine a beautiful castle had been standing for centuries. In the cellar of the castle an intricate network of webbing had been constructed by the industrious spiders who lived there. One day a great wind sprang up and destroyed the webs. Frantically the spiders worked to repair the damage. For you see, they thought it was their webbing that was holding up the castle."

Thirty years ago, this charming tale was regularly told to students specializing in logic, to leave them in no doubt as to how their field was regarded by their fellow mathematicians. Even a casual examination of Monk's book makes it apparent that nowadays the "spiders" are to be found all over the castle. Far from being obsessively concerned with providing a "secure" foundation for mathematics, logicians freely use whatever mathematics they need. For example Monk's book makes use of Lagrange's four squares theorem and the properties of real-closed fields, not to mention "large" transfinite cardinal numbers. Conversely, the methods of logic have proved useful in solving problems from various parts of mathematics. Monk discusses (without giving proofs) the negative solutions to the word problem for groups and to Hilbert's tenth problem which the development of the part of mathematical logic called recursion theory has made possible. (The main theorem on Hilbert's tenth problem should either be attributed to Matiyasevic alone, or Hilary Putnam's name should be added to Monk's list.) Although the Ax-Kochen solution to a problem of Artin and the Bernstein-Robinson solution to Halmos' invariant subspace problem are not developed in Monk's book, some of the basic methods needed (decidable theories, ultraproducts, nonstandard analysis) are included.

Have all of these developments quite erased the suspicion with which mathematical logic has been regarded in our profession? It is surely true that few departments maintain the "no logicians need apply" attitude common a quarter of a century ago. Nevertheless it is not difficult to discern much of the old prejudice just beneath the surface. How else are we to understand the negative reaction of so many analysts to Abraham Robinson's brilliant use of the methods of model theory to legitimize Leibnizian infinitesimal methods in analysis? If we boast of the simple and elegant proofs that Robinson's "nonstandard" methods make possible, we are reminded that the decisive test of new methods is their utility in obtaining new results. If however, we point out that it was by using nonstandard methods that the existence of invariant subspaces for polynomially compact operators was first demonstrated, then we are told of the simple "standard" proofs now available.

An extreme example of this negative reaction to nonstandard analysis is the remarkable treatment accorded Jerome Keisler's *Elementary calculus* in this BULLETIN. Keisler's book is an attempt to bring back the intuitively suggestive Leibnizian methods that dominated the teaching of calculus until comparatively recently, and which have never been discarded in parts of applied mathematics. A reader of Errett Bishop's review of Keisler's book would hardly imagine that this is what Keisler was trying to do, since the review discusses neither Keisler's objectives nor the extent to which his book realizes them. Bishop identifies Keisler's book with the set-theoretic excesses of the so-called new mathematics (with which it has nothing in common), objects to the use of axioms in mathematics (according to Bishop, the use of '$0 = 1$' as an axiom is only objectionable to most mathematicians because it would "make mathematics too easy, and put them out of business"), and objects to Keisler's description of the real numbers as a convenient fiction (without informing his readers of the constructivist context in which this objection is presumably to be understood).

A reader not trained in modern logic, noting the diverse character of the material treated by Monk, may well have difficulty discerning a unifying theme. In fact there is a unifying theme which makes logic a single discipline: the use of formal "languages". Such languages were first used in the pioneering work of Frege, Peano, and Whitehead and Russell as the ultimate step in the demand for ever higher standards of rigor in mathematical proofs. The very processes of logical deduction were reduced to mere combinatorial manipulation of symbols. It is of course from this side of the subject, sometimes called *proof theory*, that logic derives its name. But the languages used by logicians have uses besides making possible a systematic analysis of mathematical proof. Thus, it is inherent in the purely symbolic character of formal languages that different interpretations of the same locution are possible. Development of this theme leads to the subject called *model theory*. Studying the relationship between proof theory and model theory leads to a key result: the Gödel completeness theorem. The use of formal languages for computing makes possible a precise explication of the intuitive notion of effective computability and represents the beginning of *recursion theory*. Proof theory, model theory, and recursion theory are all involved in determining whether or not various "theories" are decidable, i.e. whether or not there is an algorithm for deciding whether or not an arbitrary sentence belongs to a given theory.

In the original developments it was very much to the point that locutions of the languages used could (at least in principle) actually be written down. However in much contemporary work (e.g. infinitary languages, generalized recursion theory) it is commonplace to work with "languages" with uncountably infinite "alphabets" or involving "expressions" of infinite length. In applications of nonstandard analysis, it is quite usual to work with languages containing "symbols" serving as "names" not only for each real number, but even for each real-valued function defined on the real numbers.

Is there any relation at all between mathematical logic as it is practiced and the "foundations" of mathematics? Logicians hardly conceive of themselves as providing the foundation without which mathematics would collapse. However the fact that all of ordinary mathematics can be captured in a single formal system, means that theorems about such a system can lead to insights about mathematics. Thus, Gödel's incompleteness theorem can be interpreted as indicating limits in principle on our ability to decide questions about elementary number theory, and even (using Matiyasevic's theorem) questions about the solvability of Diophantine equations. Techniques for constructing models of the Zermelo-Fraenkel axioms for set theory not only force one to realize that it is not so clear what it means to settle such a question as the continuum hypothesis, but also lead to new interesting axioms which can be used to settle outstanding problems in descriptive set theory.

Although Monk's book is self-contained (except for presupposing a little knowledge of set theory), it is probably best suited to a reader with some previous knowledge of the subject. The book provides a systematic introduction at a high level to a number of important topics. The author has gone to some trouble to provide really elegant treatments of a varied subject matter. On the other hand, Monk is quite uncompromising in obtaining the best known result in a number of areas even at the cost of some rather messsy details (e.g. the essential undecidability of the theory in the language of set theory whose axioms are only the existence of an empty set and the axioms of extensionality and pairs). There is an excellent collection of exercises.

The first part of the book, on recursion theory, begins with the language of Turing "machines" and develops the parts of the subject needed in the theory of undecidability, mainly the basic properties of recursively enumerable sets. Having this material available from the beginning makes it possible to be quite rigorous about computability from the first. A key role is played by Church's thesis which asserts that the recursive functions are precisely those which are intuitively effectively computable. Monk refers to the *weak Church's thesis* as the assertion that the various particular functions that arise in the work and are intuitively seen to be effectively calculable are in fact recursive. Invoking this weak Church's thesis in then just a matter of saving ourselves the trouble of some messy verifications. Although Monk indicates that he does not use the full Church's thesis, he really does implicitly use it whenever he proceeds from a theorem which asserts that some function is not recursive to the conclusion that "there is no automatic method" for computing the function, and hence in all of his undecidability results.

The second part gives the main theorems about first-order logic (completeness, compactness, Herbrand's theorem) from an "advanced" standpoint. There is a surprising amount of space devoted to sentential logic (i.e. propositional calculus). (Why should we care about the fact that three axioms can be reduced to one very complicated one? See exercise 8.50 the hint for which consists of almost an entire page of terrible-looking formulas.) Monk's own research interests are reflected in chapters on Boolean and cylindric algebras.

The third part is concerned with the decidability of theories. Decidability is proved for various theories using elimination of quantifiers (Presburger arithmetic) and model completeness (real-closed fields). Undecidability is proved for various theories using recursion-theoretic methods, especially effective inseparability and spectral representability. A particularly elegant and complete treatment is given (using Löb's theorem) of Gödel's result about the unprovability of sentences asserting the consistency of a theory in the theory itself.

Almost one third of the book (Part IV) is devoted to model theory. After two basic chapters on methods of constructing models and elementary equivalence (including the Skolem-Löwenheim theorem), there are introductions to a number of important topics including nonstandard analysis, Craig's interpolation theorem, and saturated structures. The book ends with a final part dealing briefly with a number of "unusual" logics including $L_{\omega_1\omega}$.

Monk's notation is sometimes unnecessarily heavy-minded. E.g. "$\mathbf{m}$ is a positive integer" is rendered "$\mathbf{m} \in \omega \sim 1$". Notions such as recursive enumerability and effective inseparability are never defined for sets of sentences but only for sets of natural numbers. The result is that cumbersome notation for the range of an extension of a Gödel numbering is carried along in a way which can really be a nuisance to a reader. The prose unfortunately does not share the clean elegance of the mathematical development. Finally, it is surprising that Springer-Verlag did not catch some carelessness with proper names, e.g. "Weierstrauss".

But these minor matters aside, Monk has brought together an enormous range of interesting material. He has written an important and valuable book which will be a standard reference for some time to come.

A few errata: p. 19, line 8, the subscript 0 should be 1; p. 31, line 13, "on" should be "an"; p. 81, line -9, the word "recursive" is (crucially) missing after "binary"; p. 267, in the proof of $(*)$, the list of finite structures should repeat each one infinitely often; p. 350, line 9, the subscript on *Fmla* should be $\mathfrak{L}$; p. 442, line 10, *Fmla* should have the superscript $\mathfrak{n}$ (in addition to its subscript).

MARTIN DAVIS

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 5, September 1977

Connections, curvature, and cohomology. Vol. I: *de Rham cohomology of manifolds and vector bundles*, by Werner Greub, Stephen Halperin and Ray Vanstone, Academic Press, New York and London, 1972, xix + 443 pp., \$27.50.

Connections, curvature, and cohomology. Vol. II: *Lie groups, principal bundles, and characteristic classes*, by Werner Greub, Stephen Halperin and Ray Vanstone, Academic Press, New York and London, 1973, xxi + 541 pp., \$35.00.

Connections, curvature, and cohomology. Vol. III: *Cohomology of principal bundles and homogeneous spaces*, by Werner Greub, Stephen Halperin and Ray Vanstone, Academic Press, New York, San Francisco, London, 1976, xxi + 593 pp., \$49.50.[1]

The topic of these volumes, relations between the topology and the differential geometry of manifolds, in particular, the notion of "characteristic classes", has occupied mathematicians for a long time. The first instances are probably Gauss's expression for the linking number of two curves by a double integral; and Dyck's theorem $\int_S K\,dA = 2\pi\chi_S$, where S is a closed surface, K the Gauss curvature and χ_S the Euler characteristic (1888, for a surface in 3-space; later proved (by Blaschke?) intrinsically, with Gauss's Theorema Egregium and the Gauss-Bonnet formula). The latter theorem is still *the* model for the present topic.

Another important example is Hopf's theorem $\sum j_P = \chi_M$, where the j_P are the indices of the zeroes of a vector field V on the closed manifold M, and χ_M again the Euler characteristic; there is also its earlier companion: The

[1] These books are included as volumes in the Pure and Applied Mathematics Series of Academic Press. Vol. I is numbered 47-I, Vol. II is 47-II, and Vol. III is 47-III.