CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

BY LOUIS HALLE ROWEN

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We will carry the following hypotheses throughout this paper: $F$ is a field of characteristic $\neq 2$; $A$ is a central simple $F$-algebra, i.e. a simple $F$-algebra of finite dimension, with center $F$; $A$ has an involution $(\ast)$ of first kind, i.e. an anti-automorphism of degree 2 which fixes the elements of $F$. The classic reference on central simple algebras is [1], which also treats involutions.

The dimension of $A$ (over $F$) must be a perfect square, which we denote as $n^2$. A famous conjecture is that $A$ must be a tensor product of a matrix subalgebra (over $F$) and quaternion subalgebras (over $F$); since the conjecture is easily proved when $n < 8$, the first case of interest is when $n = 8$. The main theorem of this paper is the following result when $A$ is a division algebra.

**Main Theorem.** If $n = 8$, then $A$ has a maximal subfield which is a Galois extension over $F$, with Galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The proof relies heavily on a computational result of Rowen and Schild, which will be given below. Before sketching the proof of the main theorem, we start with some general results (true for any $n$), which can be verified easily.

**Proposition 1.** Given a subfield $K$ of $A$ containing $F$, we have an involution of $A$ (of the first kind), which fixes the elements of $K$.

**Proposition 2.** Suppose $A$ is also a division algebra. Suppose $K$ is a non-maximal subfield of $A$ (containing $F$), with an automorphism $\phi$ over $F$, having degree 2. Then $\phi^*: K \to K^*$ can be given by conjugation in $A$, by an element which is symmetric (resp. antisymmetric) with respect to $(\ast)$.

Let $\overline{F}$ denote the algebraic closure of $F$, and let $M_n(\overline{F})$ be the algebra of matrices over $\overline{F}$. Then $(\ast)$ induces an involution on $M_n(\overline{F}) \approx A \otimes_F \overline{F}$, given by $(\Sigma a_i \otimes \beta_i)^* = \Sigma a_i^* \otimes \beta_i$, for $a_i \in A$ and $\beta_i \in F$. We say $(\ast)$ is of symplectic type if the extension of $(\ast)$ to $M_n(\overline{F})$ is symplectic, i.e. not cogredient to the transpose (of matrices), cf. [1, p. 155]. Such an involution exists iff $n$ is even, in which case we can build a "universal" $F$-algebra with symplectic type involution.

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tion, which we call $F(Y, Y^g)$. (See [3, §5] for details of construction, and for the properties of $F(Y, Y^g)$; here we write "$F$" in place of "$\Omega$", which is used in [3].) Let $F_1 = \text{Cent } F(Y, Y^g)$, which is a central simple $F_1$-algebra of dimension $n^2$, with symplectic-type involution. By [3, Theorem 30], we can prove our main theorem by showing (when $n = 8$) that $F(Y, Y^g)$ has a maximal subfield which is a Galois extension over $F_1$, with Galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Henceforth set $n = 8$. The key step to our main theorem is the following fact, which follows directly from a computation of Rowen and Schild [4] (done with the help of the IBM 370 computer at Bar Ilan University):

**Lemma 1.** $F(Y, Y^g)$ has an element $x$ whose square is central.

(The element $x$ is in fact given by an explicit formula.) Now we sketch the proof of Theorem 1, working in $F(Y, Y^g)$. Using Proposition 1, we can find an involution (of first kind) under which $x$ is symmetric; using Proposition 2, we can then modify the involution $(\ast)$ such that $(\ast)$ is symplectic and $x$ is antisymmetric; i.e. $x^\ast = -x$. Then there is a symmetric element $y$, such that $yxy^{-1} = -x$. If $y^2 \in F_1(x)$ then $y$ and $x$ generate a quaternion $F_1$-subalgebra invariant under $(\ast)$; in such a case, one concludes that $F(Y, Y^g)$ is a tensor product of quaternion subalgebras, and the theorem follows immediately. Thus we may assume $F_1(y^2) \cap F_1(x) = F$. Note $F_1(y^2) \neq F_1(y)$, and $y$ (being symmetric) has degree 4 over $F_1$. Hence $[F_1(y^2): F] = 2$. We can then find $z$ symmetric, such that $zxx^{-1} = x$ and $zy^2z^{-1} = -y^2 + \text{tr}(y^2)/4$. If $z^2 \in F_1(x, y^2)$ then $z^2 \in F_1(y^2)$ (since $z^2$ is symmetric and $x, xy^2$ are antisymmetric); in this case $y^2$ and $z$ generate a quaternion subalgebra invariant under $(\ast)$, and again we are done. Thus, we may assume $F_1(z^2) \cap F_1(x, y^2) = 0$. Hence $F_1(x, y^2, z^2) = F_1(z)F_1(y^2)F_1(z^2)$, and the theorem follows immediately. Q.E.D.

One interesting aspect of this theorem is that $F(Y, Y^g)$ is used to produce a positive result. Previously, universal PI-algebras (without involution) had been used by Amitsur [2] to produce an important negative result.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, RAMAT GAN, ISRAEL