In this article we announce a series of results on the existence of harmonic maps from surfaces to Riemannian manifolds and, as corollaries of these results, obtain theorems on the existence of minimal immersions of 2-spheres.

Let \( N \) be a compact connected Riemannian manifold and, for convenience, assume that \( N \) is isometrically imbedded in \( \mathbb{R}^k \) for some sufficiently large \( k \). Let \( M \) be a closed Riemann surface with any metric compatible with its conformal structure. A map \( s \in L^2(M, \mathbb{R}^k) \cap C^0(M, N) \) is called harmonic if it is an extremal map of the energy integral

\[
E(s) = \int_M |ds|^2 d\mu_M = \int_M \text{trace } I(x) d\mu_M
\]

where

\[
I(x) = \sum_{i=1}^k ds^i \otimes ds^i(x) \in T^*_x(M) \otimes T^*_x(M).
\]

Harmonic maps satisfy an Euler-Lagrange equation

\[
\Delta s + A(s)(ds, ds) = 0
\]

in a weak sense, where \( A \) is the second fundamental form of the imbedding \( N \subset \mathbb{R}^k \). It then follows from regularity theorems that harmonic maps are \( C^\infty \). If \( s \) is harmonic and a conformal immersion, it is also an extremal for the area integral.

Proving the existence of harmonic maps of \( M \) into \( N \) by direct methods from global analysis such as Morse theory or Ljusternik-Schnirelman theory applied to \( E \) defined on some function space manifold is difficult, because \( E \) is invariant under the conformal group of \( M \), and the extremal maps of \( E \) form a non-compact set when \( M = S^2 \). In particular, \( E \) does not satisfy condition C of Palais-Smale. However, for \( \alpha > 1 \), a slightly different integral,

\[
E_\alpha(s) = \int_M (1 + |ds|^2)^\alpha d\mu_M
\]
for $s \in L^2(M, R^k) \cap C^0(M, N) = L^2(M, N)$, is $C^2$ and satisfies the Palais-Smale condition C in a complete Finsler metric on $L^2(M, N)$. If we normalize the area of $M$ to be 1 then, as $\alpha \to 1$, $E_\alpha(s) \to E(s) + 1$. By examining the convergence of a sequence $s_\alpha$ of critical maps of $E_\alpha$ as $\alpha \to 1$, various results on the existence of harmonic maps are obtained.

**Main convergence theorem.** Let $s_\alpha(t)$ be a sequence of critical maps of $E_\alpha(t)$, $\alpha(t) \geq 1$, $\lim_{t \to \infty} \alpha(t) = 1$. Then there exist a subsequence $t'$, a harmonic map $s: M \to N$ and a finite number of points $\{x_1, \ldots, x_l\}$ such that $s_\alpha(t') \to s$ in $C^1(M - \{x_1, \ldots, x_l\}, N)$. Moreover, there exist $l$ nontrivial harmonic maps $\tilde{s}_k: S^2 = R^2 \cup \{\infty\} \to N$, $k = 1, 2, \ldots, l$, such that for $x \in R^2$, $\tilde{s}_k(x) = \lim_{t' \to \infty} s_\alpha(t')(x_k + \rho_{t', k} x)$ where $\lim_{t' \to \infty} \rho_{t', k} = 0$. Note that $l = 0$ is possible.

In the proof of the convergence theorem, we make use of the following extension theorem.

**Extension theorem.** Let $D$ denote the open unit disk. Let $s: D - \{0\} \to N$ be a harmonic map defined on $D - \{0\}$. If $E(s) < \infty$, then $s$ extends to a smooth harmonic map $\tilde{s}: D \to N$.

The convergence theorem is used to obtain a series of results on harmonic maps. The first result applies to the case $M \neq S^2$. Every free homotopy class in $C^0(M, N)$ induces a map from $\pi_1(M)$ into $\pi_1(N)$, which is defined only up to conjugation by an element of $\pi_1(N)$, due to the lack of base point.

**Theorem 1.** There exists a minimizing harmonic map among all maps inducing the same conjugacy class of maps from $\pi_1(M)$ to $\pi_1(N)$.

**Corollary.** If $\pi_2(N) = 0$, then there exists a minimizing harmonic map in every homotopy class of maps in $C^1(M, N)$.

In the case $\pi_2(N) \neq 0$, $\pi_1(N)$ acts on $\pi_2(N)$ by moving the base point around representatives of elements. Given an element $\Gamma$ in the free homotopy classes in $C^0(S^2, N)$, we associate with $\Gamma$ a subset $\pi_1(\Gamma) \subset \pi_2(N)$ consisting of all based homotopy classes formed by connecting the spheres in $\Gamma$ with the base point in $N$. Thus $\pi_1(\Gamma)$ is an orbit of $\pi_1(N)$ in $\pi_2(N)$.

**Theorem 2.** There exists a set of free homotopy classes $\Lambda_i \subset C^0(S^2, N)$ such that elements $\lambda_i \in \pi_1(\Lambda_i)$ generate the group ring $\pi_2(N)$, and each $\Lambda_i$ contains a minimizing harmonic map $s_i: S^2 \to N$.

In general there may be no nontrivial minimizing harmonic maps. However, we do have the following special result.

**Theorem 3.** If the universal covering space of $N$ is not contractible, then there exists at least one nontrivial harmonic map $s: S^2 \to N$. 
There is a close relationship between harmonic maps and minimal surfaces. If $U$ is an open set in $M$ and an immersion $s: U \to N$ is conformal, then $s$ is harmonic if and only if $s(U)$ is minimal. Given any harmonic map $s: M \to N$, we define

$$w(z) = |s_y(z)|^2 - |s_x(z)|^2 + 2i(s_x(z), s_y(z))$$

where $z = x + iy$ is a local isothermal coordinate chart on $M$. Let $\phi(z) = w(z)d\bar{z}^2$. From the Euler-Lagrange equations for the energy integral, one can show that if $s$ is harmonic then $\phi(z)$ is a holomorphic quadratic differential. Therefore, $\phi(z) = 0$ if $s: S^2 \to N$ is harmonic. A more general theorem applies to all surfaces $M$.

**Theorem 4.** If $s$ is a critical map of $E$, where the variation is taken over both the map $s$ and the conformal structure on $M$, then the holomorphic quadratic differential $\phi$ associated with $s$ is identically zero.

Since there is only one conformal structure on $S^2$, we obtain the result that if $s: S^2 \to N$ is harmonic, then $s$ is a minimal immersion except at points $z_0$ with $s_x(z_0) = s_y(z_0) = 0$.

**Theorem 5.** If $s: S^2 \to N$ is harmonic, then $s$ is a conformal branched immersion and $s(S^2)$ is a minimal surface except at the branch points of $s$.

**Corollary.** The maps $s_i: S^2 \to N$ in the statement of Theorem 2 can be taken to be minimal branched immersions.

**Main theorem.** Let $N$ be a $C^\infty$ compact Riemannian manifold of dimension $\geq 3$ such that the universal covering space of $N$ is not contractible. Then there exists a nonconstant $C^\infty$ map $s: S^2 \to N$ such that $s: S^2 \setminus \{x_1, \ldots, x_t\} \to N$ is a conformal minimal immersion and $x_1, \ldots, x_t$ are branch points of $s$.

**References**

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, CHICAGO CIRCLE, CHICAGO, ILLINOIS, 60680