THE EXISTENCE AND UNIQUENESS OF A SIMPLE GROUP GENERATED BY \(\{3, 4\}\)-TRANSPOSITIONS

BY JEFFREY S. LEON AND CHARLES C. SIMS

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Recently Fischer [1] discovered three finite simple groups each of which contains a conjugacy class \(D\) of involutions such that for all \(x\) and \(y\) in \(D\) the order of the product \(xy\) is 1, 2, or 3. Such a class is called a class of 3-transpositions. More generally, if \(\pi\) is a set of positive integers and \(D\) is a conjugacy class of involutions in the finite group \(G\), then \(D\) is said to be a class of \(\pi\)-transpositions in \(G\) if \(D\) generates \(G\) and for all noncommuting elements \(x\) and \(y\) of \(D\) the order of \(xy\) is in \(\pi\). Fischer has produced evidence suggesting the existence of a new simple group containing a class of \(\{3, 4\}\)-transpositions. Fischer determined a number of properties of the group, including its order, which is \(2^{41}3^{13}5^67^211 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47\) or approximately \(4.15 \times 10^{33}\). However, the questions of the existence of such a group and the uniqueness of its isomorphism type remained unanswered.

We have now constructed a simple group \(G\) having the properties specified by Fischer and in addition we have shown that \(G\) is determined, up to isomorphism, by certain of these properties. A description of the \(13,571,955,000\) \(\{3, 4\}\)-transpositions in \(G\) has been obtained and the action of a set of generators for \(G\) on these transpositions has been determined. The details of the construction and the proof of uniqueness, which involve extensive use of a computer, will appear elsewhere.

If \(H\) is any group, then \(Z(H)\) will denote the center of \(H\), \(H'\) the commutator subgroup of \(H\), and \(O_2(H)\) the largest normal 2-subgroup of \(H\). If \(h\) is an element of \(H\) and \(K\) is a subgroup of \(H\), then \(C_K(h)\) is the centralizer in \(K\) of \(h\) and \(h^K\) is the set of \(K\)-conjugates of \(h\).

Let \(L\) be a perfect 2-fold covering group of the simple group \(2E_6(2)\). That is, \(L' = L\), \(|Z(L)| = 2\), and \(L/Z(L)\) is isomorphic to \(2E_6(2)\). These conditions determine \(L\) up to isomorphism. In \(\text{Aut}(L)\) there is a unique conjugacy class of involutions \(\sigma\) centralizing a subgroup of \(L\) isomorphic to \(Z_2 \times F_4(2)\). Let \(E\) denote the split extension of \(L\) by \(\langle \sigma \rangle\) and let \(d\) generate \(Z(E)\).

The smallest of the Fischer simple groups generated by 3-transpositions has order \(2^21^37^39^52^7 \cdot 11 \cdot 13\) and is usually denoted \(F_{22}\). It is known that

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|Aut(F_{22})| = 2 and that Aut(F_{22}) is generated by a unique class of {3, 4}-transpositions. In E there are exactly two conjugacy classes of subgroups isomorphic to Aut(F_{22}), which are interchanged by an automorphism of E acting trivially on L. Fix a subgroup S of E isomorphic to Aut(F_{22}). In S' choose two noncommuting 3-transpositions d_2 and d'_2 and choose a {3, 4}-transposition d_3 in S. Set K = C_S(d_2) \cap C_S(d'_2). Then |K : K'| = 4 and K' is isomorphic to PSU(4, 3). There is a unique subgroup Q of index 2 in O_2(C_E(d_2)) which is normalized by K. Define five subgroups of E as follows

E_1 = E,
E_2 = C_E(d_2),
E_3 = C_E(d_3) \cong Z_2 \times Z_2 \times F_4(2),
E_4 = S,
E_5 = KQ.

The group E_2/\langle d \rangle is a split extension of an extra-special group of order 2^{21} by a group of the form PSU_6(2) \cdot Z_2.

We can now state our main result.

**Theorem.** There exists a simple group G, unique up to isomorphism, such that

1. |G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.
2. G contains E and C_G(d) = E.
3. Under conjugation by E, the class D = d^G breaks up into five classes D_i, 1 \leq i \leq 5.
4. For 1 \leq i \leq 5 there is an element x_i in D_i such that C_E(x_i) = E_i.

The class D is a class of {3, 4}-transpositions in G. In fact D_1 = \{d\}, D_1 \cup D_2 \cup D_3 is the set of elements in D commuting with d, and elements of D_4 and D_5 have products of order 3 and 4 with d respectively. It can be shown that x_2 = d_2 and x_3 = d_3.

**Reference**