BIFURCATION OF PERIODIC ORBITS ON MANIFOLDS, AND HAMILTONIAN SYSTEMS

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We consider a vector field $X_0$ having a whole submanifold $\Sigma \subset M$ of periodic points, and ask if any periodic orbits persist under small perturbation, i.e. do all vector fields $Y$ sufficiently near $X_0$ have periodic orbits lying near $\Sigma$. $\Sigma$ is assumed to be compact. Although in the general case there are simple counterexamples (e.g. on $\Sigma = n$ torus) some natural hypotheses on $\Sigma$ and the flow of $X_0$ guarantee periodic orbits for $Y$, which are thought of as bifurcating off the manifold $\Sigma$. Our method here is closely analogous to that of Moser [2], [3], and also his method of averaging on manifolds [1].

In the case of Hamiltonian flows, these methods take on added significance, and the classical action integral makes an appearance. Here the results may be viewed as an extension to $S^1$-actions of results of Weinstein carried out for $Z_n$-actions [4], [5].

1. The general case. Let $X_0$ be a vector field on a manifold $M$ and $\phi^t$ its induced flow. A nondegenerate periodic manifold of $X_0$ of period $r$ is a $\phi^t$-invariant submanifold of $M$ such that $\phi^r(z) = z$ for all $z \in \Sigma$, and such that 1 is an eigenvalue of $d\phi^r_z$ of algebraic multiplicity $k = \dim \Sigma$.

We denote the space of vector fields over $M$ by $X(M)$, having the usual $C^k$ norm $\|\cdot\|_k$. We parametrize a neighborhood of the identity in $\text{Diff}(M)$ by a neighborhood of $0 \in X(M)$ by taking a metric and setting $u(z) = \exp_z U(z)$, for $U \in X(M)$ small enough. We define an operator $P(u): X(M) \rightarrow X(M)$ which transports vectors at $z$ to vectors at $u(z)$ by setting, for $W \in T_z M$,

$$P(u)W = \frac{d}{dh} \bigg|_{h=0} \exp_z(U(z) + hW).$$

**Lemma A.** Let $X_0$ be a $C^{l+1}$ vector field on $M^r$ generating the flow $\phi^t$, having a compact nondegenerate periodic manifold $\Sigma$ of period 1. Suppose $Y$ is a vector field so that $\|Y - X_0\|_{l+1} < \epsilon$ in some neighborhood of $\Sigma$. Then for $\epsilon$ sufficiently small, there exists a $C^l$ vector field $V \in X(\Sigma)$, a $C^l$ embedding $u: \Sigma \rightarrow M$ near the inclusion, and $\phi^t$-invariant function $\lambda: \Sigma \rightarrow R$ so that...
(i) \( P(u)V(z) = du_2X_0 - \lambda(z)Y(u(z)) \),

(ii) \([V, X_0] = 0\),

(iii) \( \langle V, X_0 \rangle = 0 \) for \( \langle \cdot, \cdot \rangle \) a \( \phi^t \)-invariant metric on \( \Sigma \).

Writing \( u(z) = \exp_zU(z) \), \( z \in \Sigma \), let \( U_\Sigma \) = component of \( U \) tangential to \( \Sigma \). Then \( u \) and \( V \) above are unique if we impose the normalization

(iv) \( \int_0^1 d\phi^-tU_\Sigma(\phi^t(z))dt = 0 \).

In particular, if \( V(\xi) = 0 \), then \( u(\phi^t(\xi)) \) is a periodic trajectory of \( Y \) of period \( 1/\lambda(\xi) \). The proof is to solve the linearized equations and then solve the nonlinear system by iteration. The transport operator \( P(u) \) is needed not only to give equation (i) sense, but also its independence of the derivatives of \( u \) is crucial to avoid loss of derivatives in the iteration.

**Corollary.** If \( X_0 \) has a compact nondegenerate periodic manifold \( \Sigma \) of period \( \tau \), suppose that the flow \( \phi^t \) of \( X_0 \) defines a free \( S^1 \) action on \( \Sigma \). Then if the Euler characteristic \( E(\Sigma/S^1) \neq 0 \), every \( Y \) sufficiently close to \( X_0 \) has at least 1 periodic orbit near \( \Sigma \) with period close to \( \tau \).

2. The Hamiltonian case. We now consider a manifold \( P \) with a symplectic 2-form \( \Omega \), and consider Hamiltonian vector fields \( X_0, X \) with Hamiltonian functions \( H_0, H \), which are close to one another. We consider a noncritical energy surface \( E_0^c = \{ z : H(z) = c \} \) of \( H_0 \) and the corresponding energy surface \( E^c \) of \( H \). We suppose that \( X_0 \) restricted to \( E_0^c \) has a compact nondegenerate periodic manifold \( \Sigma \subset E_0^c \). In order to compare the two flows, we take a diffeomorphism \( \beta : E_0^c \rightarrow E^c \) between the two energy surfaces so that \( \beta^*X = d\beta^{-1}X(\beta(z)) \) is a vector field on \( E_0^c \) near \( X_0 \), and now apply Lemma A to get \( u(z), V(z) \) and \( \lambda(z) \) satisfying (a) \( duX_0 - \lambda\beta^*X = P(u)V \), (b) \([V, X_0] = 0\), (c) \( \langle V, X_0 \rangle = 0 \).

Assuming that \( \Omega \) induces an exact 2-form \( d\alpha \) in a neighborhood of \( \Sigma \), set \( \gamma_t = \beta \cdot u \cdot \phi^t(\xi) \) and introduce the \( X_0 \)-invariant function \( S(\xi) = \int_{\gamma_t(\xi)}\alpha \) defined on \( \Sigma \). \( S(\xi) \) is simply the classical “action” of the closed path \( \gamma_t(\xi) \) on \( E^c \). Using (a) and (b), one finds \( QV \perp \Omega = dS \), where \( Q : TM \rightarrow TM \) is a non-singular map near the identity. \( V \) is “almost” the Hamiltonian vector field of the function \( S \), and \( V(\xi) = 0 \) if and only if \( \xi \) is a critical point of \( S \).

One can replace the hypothesis that \( \Omega = d\alpha \) with the assumption that \( \Sigma \) is “exact” in the sense of Weinstein [5], [6].

In summary, the number of periodic orbits which persist on a given energy surface under Hamiltonian perturbations can be estimated by

\[ \# \text{critical points of } S \geq \text{Cat } \Sigma \]

where “Cat” is the Liusternik-Schnirelman category. However, since \( S \) is invariant under \( S^1 \)-action on \( \Sigma \) induced by \( X_0 \), the number of critical points of \( S \) can be estimated in terms of the category at the quotient space \( \Sigma/S^1 \); see Weinstein [6].
REFERENCES


