OPTIMIZATION OF THE NORM OF THE LAGRANGE INTERPOLATION OPERATOR

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On an interval \([a, b]\), we may place points \(t_0, \ldots, t_n\) such that \(a = t_0 < t_1 < \cdots < t_n = b\). Using these points, called nodes, we may construct unique polynomials \(y_0, \ldots, y_n\) of degree \(n\), such that, for \(1 \leq i, j \leq n\), \(y_j(t_i) = 1\) and \(y_i(t_j) = 0\) for \(j \neq i\). The Lagrange interpolating projection on the nodes \(t_0, \ldots, t_n\) is the operator which takes any function \(f\) continuous on \([a, b]\) to the polynomial \(\sum_{i=0}^{n} f(t_i)y_i\). It is easily seen that this projection is bounded for any degree \(n\), for any interval \([a, b]\), and for any set of nodes in \([a, b]\). The norm is easily shown to be the sup norm of \(\Lambda = \sum_{i=0}^{n} |y_i|\), called the Lebesgue function of the projection, and thus the norm depends exclusively on the placement of \(t_0, \ldots, t_n\).

It is irrelevant, in attempting to minimize the norm, to move \(t_0\) or \(t_n\). Of the function \(\Lambda\), it is true that \(\Lambda(t_i) = 1\) for \(0 < i < n\), while if \(n > 2\) and if \(t\) is not a node, then \(\Lambda(t) > 1\). Let \(\lambda_1, \ldots, \lambda_n\) be the values given by

\[
\lambda_i = \sup_{t \in [t_{i-1}, t_i]} \Lambda(t) \quad \text{for } 1 \leq i \leq n.
\]

Then \(\|\Lambda\| = \max_{1 \leq i \leq n} \lambda_i\).

It was conjectured by Serge Bernstein in 1932 that the norm of the interpolating projection is minimized when the nodes are so placed that \(\lambda_1 = \cdots = \lambda_n\), a conjecture rendered plausible, but by no means demonstrated, by the rather obvious fact that

\[
\frac{\partial \lambda_i}{\partial t_i} > 0 > \frac{\partial \lambda_{i+1}}{\partial t_i}, \quad \text{for } 1 \leq i \leq n - 1,
\]

and by the fact that moving any node into close proximity with one of its neighbors increases \(\|\Lambda\|\) without bound. This communication will give the following theorem and an outline of its proof in a series of lemmas.

**Theorem.** For any \(n \geq 2\), if the norm of the Lagrange interpolation operator on an interval \([a, b]\) with nodes \(a = t_0 < t, < \cdots < t_n = b\) is to be minimized, then it is necessary that the local maximum values \(\lambda_1, \ldots, \lambda_n\) of the Lebesgue function be equalized.

The proof of this theorem depends on the fact that \((\lambda_1, \ldots, \lambda_n)\) is a dif-

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ferentiable function of $t_1, \ldots, t_{n-1}$. For $1 \leq i \leq n$ we denote by $X_i$ the polynomial which agrees with $\Lambda$ on $(t_{i-1}, t_i)$. We denote by $\tau_i$ the (unique) point in $(t_{i-1}, t_i)$ at which $X_i(\tau_i) = \lambda_i$. It is then established that, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, $\partial \lambda_i / \partial t_j = -y_j(\tau_i)X'(t_i)$. Our theorem follows if we can show that, for any position of nodes, every $n-1 \times n-1$ submatrix of $(\partial \lambda_i / \partial t_j)_{ij}$ is nonsingular ($1 \leq i \leq n$ and $1 \leq j \leq n-1$). Using the above expression for $\partial \lambda_i / \partial t_j$, we may with artful cancellations reach the equivalent matrix $(q_i(t_j))_{ij}$, for $1 \leq i \leq n, 1 \leq j \leq n-1$, where $q_i(t) = X'(t)/(t - \tau_i)$ is a polynomial of degree $n-2$ or less.\(^1\) Thus each $n-1 \times n-1$ submatrix of $(q_i(t_j))_{ij}$ is nonsingular if and only if any $n-1$ of $q_1, \ldots, q_n$ form a basis for the space of polynomials of degree $n-2$ or less. In a succession of lemmas, we show that any $n-1$ of $q_1, \ldots, q_n$ are indeed a basis. Here, for reasons of conserving space, the proofs are not given in full.

**LEMMA 1.** The polynomials $X_1$ and $X_n$ have their full complement of $n$ roots on $[a, b]$, and $X_1'$ and $X_n'$ have their full complement of $n-1$ roots on $[a, b]$. For each $i, 2 \leq i \leq n-1$, the polynomial $X_i$ has exactly $n-1$ roots on $[a, b]$, and $X_i'$ has at least $n-2$ roots on $[a, b]$. Each root of each of the above polynomials has multiplicity one, and each root of $X_i$, $1 \leq i \leq n$, is a local extremum of $X_i$.

**PROOF OF LEMMA 1.** An obvious counting argument suffices.

**LEMMA 2.** For $2 \leq i \leq n$, $X_{i-1}'$ and $X_i'$ have no common root, nor do $X_i'$ and $X_n'$.

**PROOF.** One uses $X_{i-1}' + X_i = 2y_{i-1}'$ and an analogous identity for $X_1$ and $X_n'$. If for example $X_{i-1}'$ and $X_i'$ have a common root, then so does $X_{i-1}' + \alpha y_{i-1}'$, for any real $\alpha$. One then investigates the number of roots of $X_{i-1}' + \alpha y_{i-1}'$, when $\alpha$ is chosen so that $(X_{i-1}' + \alpha y_{i-1}')(r) = 0$, at an assumed common root $r$ of $X_{i-1}'$ and $X_i'$.

**LEMMA 3.** All roots of $X_1'$ and $X_n'$ lie on $[\tau_1, \tau_n]$.

**PROOF.** Since $X_1'$ and $X_n'$ have no common root, one can investigate convenient extreme configurations of $t_1, \ldots, t_{n-1}$.

**LEMMA 4.** The roots of $X_1'$ and $X_n'$ alternate as we pass from $\tau_1$ to $\tau_n$.

**PROOF.** Similar to Lemma 3.

**LEMMA 5.** Between $\tau_1$ and the first root of $X_{n-1}'$ appearing on $[\tau_1, \tau_n]$, there is a root of $X_n'$. The symmetric statement about $\tau_n, X_2', X_1$ also holds.

**PROOF.** Similar to Lemma 3.

**LEMMA 6.** On the interval $[\tau_1, \tau_n]$ the roots of $X_1', \ldots, X_n'$ lie in the

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\(^1\) Private communication from Dietrich Braess.
pattern $\hat{1}, n, n - 1, \ldots, 3, \hat{2}, 1, n, \ldots, 4, \hat{3}, 2, 1, n, \ldots, 1, n, n - 1, n - 2, \ldots, 1, \hat{n}$, when a number $i$ denotes a root of $X_i$, $1 \leq i \leq n$, and where $\hat{i}$ denotes the point $\tau_i$, $1 \leq i \leq n$.

**Proof.** This proof uses the method of Lemmas 3, 4, 5, and the results of all previous lemmas, and the method of induction.

**Corollary of Lemma 6.** The roots of $q_1, \ldots, q_n$ lie in the same locations as those of $X'_1, \ldots, X'_n$, save that $\tau_1, \ldots, \tau_n$ are removed from the list.

**Proof.** Clear.

**Lemma 7.** Without doing harm, we may assume that $q_j(\tau_1) > 0$ for $1 \leq i \leq n$. Under this convention, we have $\text{sgn } q_j(\tau_i) = \text{sgn } q_1(\tau_i)$ and $\text{sgn } q_j(\tau_i) = -\text{sgn } q_1(\tau_i)$ for $j \neq i$, where $2 \leq i, j \leq n$.

**Proof.** Follows easily from Lemma 6 and its Corollary.

**Lemma 8.** The set $\{q_1, \ldots, q_n\} \sim \{q_k\}$ is linearly independent for any choice of $k$, $1 \leq k \leq n$.

**Proof.** We assume the existence of a nontrivial linear combination $\alpha_1 q_1 + \cdots + \alpha_n q_n = 0$, in which, for some $k$, $2 \leq k \leq n$, we have assumed $\alpha_k = 0$, and also $\alpha_1 \geq 0$. The sets $N = \{j | \alpha_j < 0 \text{ and } 2 \leq j \leq n\}$ and $P = \{j | \alpha_j > 0 \text{ and } 2 \leq j \leq n\}$ are shown nonempty, and the polynomial $N = \alpha_1 q_1 + \sum_{j \in N} \alpha_j q_j$ is shown to alternate sign at successive points $\tau_i$, $1 \leq i \leq n$, thus having at least $n - 1$ roots, while having degree no more than $n - 2$. The contradiction implies the lemma.

Our theorem now follows from Lemma 8, in light of previous discussion.

**Added May 18, 1977.** In the paper which gives details of the above theorem, I complete the proof of Bernstein's conjecture by proving that there is a unique choice of nodes which equalizes the $\lambda_i$. I have been informed by C. de Boor and A. Pinkus that they too, using the results of the present note, have completed the proof of Bernstein's conjecture.

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