DETERMINATION OF THE AUGMENTATION TERMINAL  
FOR FINITE ABELIAN GROUPS  
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Let \( G \) be a finite abelian group, and let \( IG \) denote the augmentation ideal in the integral group ring \( ZG \). The graded ring associated with the filtration on \( ZG \) determined by the powers of \( IG \) is

\[
\text{gr } ZG = \bigoplus_{n \geq 0} IG^n/IG^{n+1}.
\]

We write \( Q_n G = IG^n/IG^{n+1} \). As is well known [1], [6], the sequence \( Q_n G \) becomes stationary after a finite number of steps. We call its terminal value the augmentation terminal, \( Q_\infty G \). We outline here a method for investigating \( Q_\infty G \) for any \( G \).

An obvious splitting allows us to assume that \( G \) is a \( p \)-group.

We choose a generator for each cyclic direct factor of \( G \). Let \( \Gamma \) be our set of such generators, and let \( \Lambda = \{ \lambda | \lambda + 1 \in \Gamma \} \). Generalizing Lemma 2 of [3] we have

**Lemma.** For \( n \geq 1 \) the set of \( n \)-fold products of elements of \( \Lambda \) generates \( IG^n \); a fortiori it generates \( Q_n G \).

If \( \lambda \in \Lambda \) there is an integer \( r \) such that \( (\lambda + 1)^p^r - 1 = 0 \). Furthermore, by the structure of \( G \), these equations are the only possible source of relations among the elements of \( \Lambda \). Hence we have immediately

**Theorem 1.** Let \( f(\lambda_1, \ldots, \lambda_k) \) be a nontrivial relator in \( Q_n G \), where the \( \lambda_i \in \Lambda \). Let \( \lambda_i + 1 \) be of order \( p^r \) in \( G \), each \( i \). Let \( X_1, \ldots, X_k \) be indeterminates over \( Z \). Then there are polynomials \( h_i(X_1, \ldots, X_k) \) with integer coefficients such that

\[
f(X_1, \ldots, X_k) - \sum_{i=1} f(X_i + 1)^{p^{r_i}} - 1 h_i(X_1, \ldots, X_k)
\]

has no terms of degree \( < n + 1 \).

Actually using this result to find relators is far from easy, as the references show [2], [7], [8]. If \( G \) is an elementary \( p \)-group we have

**Theorem 2.** The relators in \( Q_\infty G \) are generated by \( \{ p\lambda | \lambda \in \Lambda \} \) and \( \{ \lambda^p \mu - \lambda \mu^p | \lambda, \mu \in \Lambda \} \).

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PROOF. Easy hand calculation shows that these are indeed relators. It is trivial to work out the group given by these relators and then use Theorem 5 of [6] (see also Theorem 3 below).

By much more tedious calculation one can show

**Lemma.** If $\lambda, \mu \in \lambda$ and $\lambda + 1, \mu + 1$ have order $p^2$ in $G$, then $\lambda^p \mu^p - \lambda^p \mu^p$ is a relator.

The appropriate generalization is readily conjectured, but a direct proof is likely to be very difficult.

We developed in [3], [7] and [8] a technique of "standard forms" which is generally suitable for determining the structure of $\mathbb{Q}^n$ modulo a given set of relators of $\mathbb{Q}_n G$. This technique may be applied to any $G$. If the order of the group so determined is the same as that of $\mathbb{Q}_\infty G$, then $\mathbb{Q}_\infty G$ has been found. Otherwise, another relator must be hunted down.

The order of $\mathbb{Q}_\infty G$ is calculated via the module index $[\mathcal{B} \cap \mathbb{Q} : \mathcal{I} G : \mathcal{B} \cdot \mathcal{I} G]$ where $\mathcal{B}$ is the maximal order in $\mathbb{Q} G$. This follows from [6], where we used our results on invertible powers of ideals [4], [5]. Specifically, by direct calculation from Theorem 5 of [6].

**Theorem 3.** Let $G$ be the direct product of $a_i$ cyclic groups of order $p^i$, $1 \leq i \leq m$. Then the order of $\mathbb{Q}_\infty G$ is $p^J$, where

$$p^J = p^{t_1} + \cdots + p^{t_m + 1} - 1$$

in which

$$t_i = a_1 + 2a_2 + \cdots + (i - 1)a_{i-1} + i(a_i + \cdots + a_m - 1), \quad 1 \leq i \leq m.$$