ON THE THEORY OF $\Pi^1_3$ SETS OF REALS

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1. An ordinal basis theorem. Assuming that $\forall x \in \omega^\omega$ ($x^\#$ exists), let $u_\alpha$ be the $\alpha$th uniform indiscernible (see [3] or [2]). A canonical coding system for ordinals $< u_\omega$ can be defined by letting $W_{0^\omega} = \{ w \in \omega^\omega : w = (n, x^\#), \text{ for some } n \in \omega, x \in \omega^\omega \}$ and for $w = (n, x^\#) \in W_{0^\omega}$, $|w| = \tau_n^L[x](u_1, \ldots, u_{k_n})$, where $\tau_n$ is the $n$th term in a recursive enumeration of all terms in the language of $ZF + V = L[\check{\alpha}], \check{\alpha}$ a constant, taking always ordinal values. Call a relation $P(\xi, x)$, where $\xi$ varies over $u_\omega$ and $x$ over $\omega^\omega$, $\Pi_1^1$ if $P^*(w, x) \Leftrightarrow w \in W_{0^\omega} \land P(|w|, x)$ is $\Pi_1^1$. An ordinal $\xi < u_\omega$ is called $\Delta_1^1$ if it has a $\Delta_1^1$ notation i.e. $\exists w \in W_{0^\omega} (w \in \Delta_1^1 \land |w| = \xi)$.

**Theorem 1** ($ZF + DC + \text{Determinacy (}\Delta_2^1\))$. Every nonempty $\Pi_3^1$ subset of $u_\omega$ contains a $\Delta_3^1$ ordinal.

**Corollary 2** ($ZF + DC + \text{Determinacy (}\Delta_2^1\))$. $\Pi_3^1$ is closed under quantification over ordinals $< u_\omega$ i.e. if $P(\xi, x)$ is $\Pi_3^1$ so are $\exists \xi P(\xi, x), \forall \xi P(\xi, x)$.

**Corollary 3** ($ZF + DC + AD$). The class of $\Pi_3^1$ sets of reals is closed under $< \delta_3^1$ intersections and unions.

Martin [3] has proved the corresponding result for $\Delta_3^1$.

2. A Kleene theory for $\Pi_3^1$. Kleene has characterized the $\Pi_1^1$ relations as those which are inductive (see [7]) on the structure $< (\omega, <) = Q_1$. Let $i_m: u_\omega \rightarrow u_\omega, m \geq 1$, be defined by letting

$$i_m(u_i) = \begin{cases} u_i, & \text{if } i < m, \\ u_{i+1}, & \text{if } i \geq m, \end{cases}$$

and then

$$i_m(\tau_n^L[x](u_1, \ldots, u_{k_n})) = \tau_n^L[x](i_m(u_1) \ldots i_m(u_{k_n})).$$

Let $R$ be the relation on $u_\omega$ coding these embeddings, i.e.

$$R = \{(m, \alpha, \beta) : m \in \omega \land \alpha, \beta < u_\omega \land i_m(\alpha) = \beta\}.$$

Put $Q_3 = \langle u_\omega, <, R \rangle$. 

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THEOREM 4. ($\text{ZF} + \text{DC} + \text{DETERMINACY} (\Delta_2^1))$. A set of reals is $\Pi_3^1$ iff it is absolutely inductive on the structure $Q_3$.

In the second part of the above characterization a relation on reals is viewed as a second order relation on $u_\omega$ and absolutely inductive means that only parameters from $\omega$ are allowed in the definitions (see [7]).

It should be mentioned here that $Q_3$ is up to absolute hyperelementary equivalence the same as $\langle u_\omega, <, T^2 \rangle$, where $T^2$ is the tree (on $\omega \times u_\omega$) coming from the Martin and Solovay [4] analysis of $\Pi_2^1$ sets (see [3] for the definition of $T^2$).

One also obtains the analog for $\Pi_3^1$ of the Souslin-Kleene representation of $\Pi_1^1$ sets in terms of well-founded trees.

THEOREM 5 ($\text{ZF} + \text{DC} + \text{DETERMINACY} (\Delta_2^1))$. A set of reals $P$ is $\Pi_3^1$ iff there is a tree $T$ on $\omega \times u_\omega$ which is recursive in the structure $Q_3$ and $P(x) \iff T(x)$ is well founded.

For the notation see [2]. The fact that every $\Pi_3^1$ set can be so represented is a well-known result of Martin and Solovay [4], the converse being new here.

Let $Q_3 = \langle u_\omega, <, \{u_n\}_{n<\omega} \rangle$. Then we also have the context of full $AD$, in which case $u_n = \mathbb{R}_n$, $\forall n \leq \omega$.

THEOREM 6 ($\text{ZF} + \text{DC} + AD$). A set of reals is $\Pi_3^1$ iff it is $\Pi_1^1$ on the structure $Q_3$.

3. Explaining the $Q$-theory. The results in §2 provide a nice explanation for the $Q$-theory (see [5], [1]) at level 3, which accounts for the structural differences between $\Pi_3^1$ and $\Pi_1^1$ sets. For example, a real is $\Delta_3^1$ iff it is absolutely hyperelementary on $Q_3$ while it is in $Q_3$ iff it is hyperelementary (i.e. parameters $< u_\omega$ are allowed) on $Q_3$. Also if $y_0$ is the first nontrivial $\Pi_3^1$ singleton then $y_0$ is hyperelementary-in-$Q_3$ equivalent to the complete inductive-in-$Q_3$ subset of $u_\omega$.

4. Higher level analogs of $L$. Assuming Projective Determinacy (PD), let $T^3$ be the tree (on $\omega \times \delta_3^1$) associated with an arbitrary $\Pi_3^1$-scale on a complete $\Pi_3^1$ set (see [6] and [2]). Let also $C_4$ be the largest countable $\Sigma_4^1$ set. The next result proves a conjecture of Moschovakis and shows that $L[T^3]$ is a correct higher level analog of $L$ for level 4.

THEOREM 7 ($\text{ZF} + \text{DC} + \text{DETERMINACY} (L[\omega^\omega] \cap \text{power} (\omega^\omega))$). For any $T^3$ as above, $L[T^3] \cap \omega^\omega = C_4$. In particular $L[T^3] \cap \omega^\omega$ is independent of the tree $T^3$.

Open problem. Is $L[T^3]$ independent of $T^3$?

Further applications of the methods developed here to the theory of $\Pi_3^1$ sets as well as details and proofs of the results announced here will appear elsewhere.
REFERENCES


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