EXISTENCE AND APPLICATIONS OF REMOTE POINTS

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All spaces are completely regular, $X^*$ is $\beta X - X$.

A point $p$ of $X^*$ will be called a remote point of $X$ if $p \notin \text{Cl}_{\beta X} D$ for every nowhere dense $D \subseteq X$. Fine and Gillman, [FG] showed that $\mathbb{Q}$, the rationals, and $\mathbb{R}$, the reals, have remote points if $CH$ holds; their proof shows that $X$ has remote points if $X$ is separable and not pseudo-compact. We prove the existence of remote points without assuming additional set theoretic axioms, under slightly stronger conditions on $X$.

Recall that a $\pi$-base (or pseudo-base) for a space $X$ is a family $\mathcal{B}$ of non-empty open sets such that every nonempty open set of $X$ includes a member of $\mathcal{B}$. The $\pi$-weight of a space is the smallest cardinality for a $\pi$-base.

**Theorem A.** If $X$ is a nonpseudocompact space with countable $\pi$-weight, then $X$ has $2^\mathfrak{c}$ remote points.

I originally proved this only for $X = \mathbb{Q}$, improving a technique from [vD1]. I am indebted to Mary Ellen Rudin for showing me how to make my ideas work for $X = \mathbb{R}$. The above theorem is a further improvement.

For applications we need a “pointed” version of extremal disconnectedness.

**Definition.** If $p \in X$, then $X$ is called extremally disconnected at $p$ if for all disjoint open $U, V \subseteq X$, $p \notin U \cap V$.

One can show that $\beta X$ is extremally disconnected at every remote point of $X$. Without much effort one deduces the following theorem. ($X$ is nowhere locally compact if no point has a compact neighborhood.)

**Theorem B.** Let $X$ be a nonpseudocompact space with countable $\pi$-weight.

(a) $\beta X$ is extremally disconnected at some point of $X^*$.

(b) If $X$ is nowhere locally compact, $X^*$ is extremally disconnected at some point.

Frolík, [F], proved that $X^*$ is not homogeneous if $X$ is not pseudocompact. Theorem B can be used to show why $X^*$ is not homogeneous, for suitable $X$.

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THEOREM C. Let $X$ be a nowhere locally compact nonpseudocompact separable first countable space. Then $X^*$ is not homogeneous because $X^*$ is extremally disconnected at some but not at all points.

This applies if e.g. $X = \mathbb{Q}$, or $X = \{\text{irrationals}\}$, or $X = \{\text{Sorgenfrey line}\}$. As another application we show that certain spaces cannot be factored as a product of spaces without isolated points. The key observation is that $X \times Y$ is not extremally disconnected at any point if $X$ and $Y$ are separable spaces without isolated points.

THEOREM D. $Y$ is not the product of two spaces without isolated points in each of the following two cases:

(a) $Y = \beta X$ for some nonpseudocompact $X$ with countable $\pi$-weight;
(b) $Y = X^*$ for some nowhere locally compact nonpseudocompact $X$ with countable $\pi$-weight.

(Actually one does not need the condition on the $\pi$-weight in (a), see [vD$_2$], but I do not know if it can be avoided in (b).)

COROLLARY. $\mathbb{Q}^*$ and $(\mathbb{Q}^*)^\kappa$ are not homeomorphic, for $\kappa \geq 2$.

I do not know if $(\mathbb{Q}^*)^2$ and $(\mathbb{Q}^*)^3$ are nonhomeomorphic.

As yet another application we mention the following curiosities.

EXAMPLE. There is an extremally disconnected space which has a connected compactification.

Indeed, if $X$ is any connected nowhere locally compact separable metrizable space, like $\mathbb{R}^\omega$, then the subspace $E$ of all points at which the connected space $\beta X$ is extremally disconnected turns out to be dense in $\beta X$, but then $E$ is extremally disconnected.

For the other application, recall that a space is called $\omega$-bounded if every countable subset has compact closure.

THEOREM E. $\mathbb{R}^*$ is the union of three pairwise disjoint dense $\omega$-bounded subspaces.

If one calls a point $p$ of $X^*$ a far point of $X$ if $p \notin \text{Cl}_{\beta X} D$ for every closed discrete subset $D$ of $X$, [vD$_1$], then the three subspaces are the remote points of $\mathbb{R}$, the far points of $\mathbb{R}$ which are not remote and the points of $\mathbb{R}^*$ which are not even far. Under CH there is a family of $2^c$ such subspaces, [W].

ADDED IN PROOF: If $\kappa$ and $\lambda$ are cardinals with $\kappa > \lambda > 1$, then $(\mathbb{Q}^*)^\kappa$ and $(\mathbb{Q}^*)^\lambda$ are not homeomorphic. The nontrivial proof will appear elsewhere.

REFERENCES


[vD$_2$] ————, When $\Pi\beta$ and $\beta \Pi$ are homeomorphic (to appear).


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