HOMOTOPY RIGIDITY OF LINEAR ACTIONS:
CHARACTERS TELL ALL

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Our aim is to present a striking rigidity phenomenon in unitary representations of compact groups. Let $U = U(n)$ be a unitary group and $H$ a closed subgroup of $U$. The homogeneous space $U/H$ is a smooth manifold with a smooth action $\lambda$ of $U$ induced by left multiplication. If $\alpha: G \to U$ is a representation of the compact group $G$, then $\lambda \circ (\alpha \times 1): G \times U/H \to U/H$ is an action of $G$ on $U/H$, and we denote this $G$-structure by $(U/H, \alpha)$. Such actions of $G$ on $U/H$ are called linear actions. We shall give a complete description of the $G$-homotopy types of linear actions on $U/H$ for a certain class of $H$. To motivate our results we shall first examine some obvious $G$-equivalences of linear actions.

If $X$ is a $U$-space, then the set of $U$-maps $\text{Map}_U(U/H, X)$ is in one-to-one correspondence with elements $x \in X$ such that $U_x \supseteq H$, where $U_x = \{u \in U|ux = x\}$ is the isotropy group of the action at $x$. For example, if $a \in U$ then the element $aH$ in $U/H$ has isotropy group $aHa^{-1}$ and the $U$-map $f: U/aHa^{-1} \to U/H$ given by $f(uaH) = uaH$ is a $U$-equivalence. Indeed if $H$ and $K$ are closed subgroups of $U$ then $U/H$ and $U/K$ are $U$-equivalent if and only if $K = aHa^{-1}$ for a suitable $a \in U$. Suppose $\alpha, \gamma: G \to U$ are representations such that there exists an $a \in U$ such that $\gamma(g) = \alpha(a^1(g)a^{-1}$ for all $g \in G$ (we say that $\gamma$ is similar to $\alpha$). The map $k: (U/H, \alpha) \to (U/H, \gamma)$ given by $k(uH) = auH$ is a $G$-equivalence. Indeed, $k$ is the composition of the $G$-equivalence $(U/H, \alpha) \to (U/aHa^{-1}, \gamma)$ induced by conjugation with $a$ in $U$ and the $U$-equivalence (hence $G$-equivalence!) $f: (U/aHa^{-1}, \gamma) \to (U/H, \gamma)$. Thus similarity of representations gives us $G$-equivalences of the associated linear actions on $U/H$. Here is another obvious way of obtaining $G$-equivalences: let $c: U \to U$ be conjugation by unitary matrices $c(a) = \bar{a}$; then if $c(H) = H$, we obtain a $G$-equivalence $c: (U/H, \alpha) \to (U/H, \bar{a})$ where $\bar{a} = c \circ \alpha$ is the representation conjugate to $\alpha$.

It is too much to hope that $(U/H, \alpha)$ is $G$-homotopy equivalent to $(U/H, \beta)$ if and only if $\beta$ or $\bar{\beta}$ is similar to $\alpha$. For example, if $H$ is a subgroup of maximal rank in $U$ and $C$ is the center of $U$ then $C \subseteq H$ and $C$ acts trivially on $U/H$, so if we let $P(U) = U/C$ be the projective unitary group (with $q: U \to P(U)$ the quotient map), then the standard left action $\lambda$ of $U$ on $U/H$ induces an action of $P(U)$ on $U/H$, and it is the similarity class of the projective representation $q \circ \alpha: G \to P(U)$ which matters. We have: if $\alpha, \beta: G \to U$ are representations and $\chi: G \to S^1 = C$ is a homomorphism such that $\beta$ or $\bar{\beta}$ is similar to $\chi \alpha$ then $(U/H, \alpha)$ is $G$-equiv-
lent to \((U/H, \beta)\), and indeed through a map which is induced by an \(R\)-linear map of \(R^{2n}\) (the underlying real vector space of the complex vector space \(C^n\) on which \(U = U(n)\) acts in the standard way). The reader would expect to find more \(G\)-equivalences of linear actions if we drop linearity, and yet more \(G\)-homotopy equivalences. The surprise is that if we make a mild restriction on \(H\) then we find that linear actions of \(G\) on \(U/H\) are rigid under homotopy: \((U/H, \alpha)\) is \(G\)-homotopy equivalent to \((U/H, \beta)\) if and only if they are \(G\)-equivalent through an \(R\)-linear map. Here is a sample result:

**Theorem 1 (Homotopy Rigidity of Linear Actions).** If \(H\) is a subgroup of \(U = U(n)\) conjugate to \(U(n - k) \times T^k\), where \(T^k\) is the \(k\)-torus and \(n > 2k\), \(\alpha, \beta: G \to U\) representations of a compact group \(G\), then a \(G\)-map \(f: (U/H, \alpha) \to (U/H, \beta)\) exists with \(f: U/H \to U/H\) a homotopy equivalence if and only if there is a linear character \(\chi: G \to S^1\) and \(\beta\) or \(\bar{\beta}\) is similar to \(\chi\alpha\).

We should point out that the condition \(n > 2k\) is not necessary: for example, homotopy rigidity of linear actions holds for \(U(5)/U(2) \times T^3\) and \(U(6)/U(2) \times T^4\), but the proof is much more involved. Similarly, the condition that \(H\) be conjugate to \(U(n - k) \times T^k\) is too strong: in [13] we show homotopy rigidity of linear actions on \(U(n + m + 1)/U(m) \times U(n) \times U(1)\) for \(mn > m + n + 1\). The right level of generality for our current approach seems to be the following: let us call a subgroup \(H\) of \(U\) friendly if \(H\) is closed, connected, of maximal rank in \(U = U(n)\) and there exists a nonzero vector \(v \in C^n\) such that \(hv = \lambda(h)v\) for some linear character \(\lambda: h \to S^1\); indeed we assume \(H\) is conjugate to a subgroup \(U(n_1) \times \cdots \times U(n_k) \subset U(n)\) with \(n_1 > \cdots > n_k = 1\) and \(n_1 + \cdots + n_k = n\) (see Borel and Siebenthal [7]). We shall outline a strategy for proving

**Conjecture A.** If \(H\) is a friendly subgroup of \(U\) then linear actions of a compact group \(G\) on \(U/H\) are rigid under homotopy.

Indeed one can conjecture that linear actions of \(G\) are rigid for \(U/H\) where \(H\) is connected of maximal rank. This is work in progress with Wu-Yi Hsiang.

An immediate consequence of our homotopy rigidity result is that the \(G\)-homotopy type of \((U/H, \alpha)\) can be read off from the character table of \(G\) (characters tell all). For example, if \(\alpha, \beta: G \to U\) are representations and \(|\text{Trace } \alpha(g)| \neq |\text{Trace } \beta(g)|\) for some element \(g \in G\), then \((U/H, \alpha)\) and \((U/H, \beta)\) have distinct \(G\)-homotopy types. An example of such a situation is given by the alternating group on five letters \(A_5\); let \(\alpha\) and \(\beta\) be the distinct irreducible 3-dimensional unitary representations,

\[
\text{Tr } \alpha(g) = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \text{Tr } \beta(g) = \frac{1 - \sqrt{5}}{2},
\]

so \((U/H, \alpha)\) and \((U/H, \beta)\) are not \(A_5\)-homotopy equivalent for any friendly subgroup \(H\) of \(U = U(3)\). Here, of course, there are no nontrivial linear characters and all characters of \(A_5\) take real values, so two linear actions \((U/H, \gamma), (U/H, \delta)\) of \(A_5\) on \(U/H\) (with \(H\) a friendly subgroup of \(U\)) are \(A_5\)-homotopy equivalent if and only if \(\gamma\) is similar to \(\delta\). The case of \(\alpha\) and \(\beta\) is especially interesting since there is an outer automorphism \(\varphi: A_5 \to A_5\) with \(\varphi^2 \alpha = \beta\). Even the cyclic group of order two \(G = Z/2Z\) gives entertaining examples: if we let \(1\) denote the trivial representation of \(G\) then there exist linear actions \(\alpha, \beta, \gamma, \delta\) on \(CP^n\) such that \((CP^n, \alpha) \approx (CP^n, \beta)\) but \((CP^n, \alpha) \not\approx (CP^n, \gamma)\).
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+ 1) \approx (CP^n, \beta + 1)$, and $(CP^n, \gamma + 1) \approx (CP^n, \delta + 1)$ but $(CP^n, \gamma) \approx (CP^n, \delta)$, where we have used \( \approx \) to indicate \( Z/2Z \)-homotopy equivalence.

In Theorem 1, \( f \) is not assumed to be a \( G \)-homotopy equivalence, that is, although there is a homotopy inverse \( f' \): \( U/H \rightarrow U/H \), we are not saying that such an \( f' \) can be found which is a \( G \)-map \( f' \): \( (U/H, \beta) \rightarrow (U/H, \alpha) \). Indeed, Pétrie [15] exhibits a \( G \)-space \( Y \), a linear action \( y \) on \( CP^k \) and a \( G \)-map \( h: Y \rightarrow (CP^k, y) \) which is a homotopy equivalence such that the induced map is equivariant \( K \)-theory

\[ h^*: K^G_*(CP^k, y) \rightarrow K^G_*(Y) \]

is not an isomorphism—this means that although \( h \) is a homotopy equivalence it is not a \( G \)-homotopy equivalence. Our approach is based on the fact that this sort of pathology cannot occur if \( Y \) is a complex projective space with a linear action (see [11]): given a \( G \)-map \( h: (CP^n, \alpha) \rightarrow (CP^n, \beta) \) such that \( h: CP^n \rightarrow CP^n \) is a homotopy equivalence, there exists an \( R \)-linear \( G \)-equivalence \( k: (CP^n, \alpha) \rightarrow (CP^n, \beta) \) such that \( h^* = k^* \) (so, in particular, \( h^* \) is an isomorphism).

This report is organized as follows: in the second section, we present an exact sequence on Picard groups of \( G \)-line bundles and popularize some work of Graeme Segal [19] on cohomology of topological groups. In the third section we examine the case \( U/H = CP^n \) and show how equivariant \( K \)-theory allows us to prove the homotopy rigidity theorem for this case. We also examine the general case of \( H \) a friendly subgroup of \( U \) and show how a result on cohomology automorphisms of \( U/H \) implies the homotopy rigidity theorem. The fourth section is devoted to proving the result on automorphisms of \( H^*(U/H, Z) \), where \( H \) is as in Theorem 1.

A few words about the background of the problem. There is an extensive literature about \( G \)-maps of spheres with linear action: de Rham [16], Atiyah and Tall [5], Lee and Wasserman [10], Meyerhoff and Pétrie [14]. The current project is the result of numerous consultations with Ted Pétrie. Thanks also go to J. F. Adams, J. Dupont, H. Glover, W.-Y. Hsiang, P. Landrock, I. Madsen, G. Segal, R. Stong and J. Tornehave for their helpful comments.

2. An exact sequence of Picard groups. Let \( X \) be a \( G \)-space, \( \text{Pic}_G(X) \) the set of isomorphism classes of complex \( G \)-line bundles over \( X \). We give \( \text{Pic}_G(X) \) the structure of a group by using the tensor product of line bundles as multiplication. If \( X \) is a CW complex, then \( H^1(X; Z) \approx [X, S^1] \) and

\[ H^2(X; Z) = [X, CP^\infty] \approx \text{Pic}_E(X), \]

where \( E \subset G \) is the subgroup consisting of the identity element.

**Theorem 2.** If \( X \) is a nonempty connected \( G \)-space and \( H^1(X; Z) = 0 \) then the following sequence is exact:

\[ \text{Pic}_G(*) \rightarrow \text{Pic}_G(X) \rightarrow \text{Pic}_E(X), \]

where \( c: X \rightarrow * \) is the collapsing map onto a point, \( i: E \subset G \) the inclusion of the identity subgroup.

**Proof.** We shall use the technique of Segal's cohomology of groups [19]: if \( A \) is an abelian \( G \)-group (\( G \) compact, \( A \) has the compactly generated
topology) then cohomology groups $H^i_G(A)$ are defined for all $i > 0$. The group $H^0_G(A)$ is the quotient of the group of all crossed homomorphisms $\varphi: G \to A$ (functions which satisfy $\varphi(gg') = \varphi(g) + g \cdot \varphi(g')$ for all $g, g'$ in $G$) modulo principal crossed homomorphisms (those which have the form $\varphi(g) = g \cdot a - a$ for a fixed $a$ in $A$). The pleasant thing about Segal's cohomology is that a short exact sequence $0 \to A' \to A \to A'' \to 0$ (meaning that $A$ is a principal $A'$-bundle with $A''$ as base) produces a long exact sequence

$$\ldots \to H^i_G(A) \to H^i_G(A'') \to H^{i+1}_G(A') \to H^{i+1}_G(A) \to \cdots$$

If $V$ is a vector space over $R$ then $H^i_G(V) = 0$ for all $i > 0$. Given our CW space $X$ we first notice that $H^i_G(\text{Map}(X, S^1))$ is precisely the set of isomorphism classes of $G \times S^1$-structures on the projection $\pi_1: X \times S^1 \to X$, that is, $H^i_G(\text{Map}(X, S^1)) = \text{Ker} \ i^!$. Since $H^1(X; Z) = 0$ we obtain an exact sequence

$$0 \to \text{Map}(X, Z) \to \text{Map}(X, R) \to \text{Map}(X, S^1) \to 1,$$

$\text{Map}(X, Z) = Z$ since $X$ is connected, and the collapsing map $c: X \to *$ induces a map of exact sequences

$$0 \to Z \to R \to S^1 \to 1$$

which in turn induces maps of long exact sequences of cohomology groups. We have

$$H^1_G(R) \to H^1_G(S^1) \xrightarrow{\delta} H^2_G(Z) \to H^2_G(R)$$

$$H^1_G(V) \to H^1_G(\text{Map}(X, S^1)) \xrightarrow{\delta'} H^2_G(Z) \to H^2_G(V)$$

where $V = \text{Map}(X, R)$ is a vector space over $R$, so in both exact sequences the extreme terms are zero, hence $\delta$ and $\delta'$ are isomorphisms; thus $c^*: H^1_G(S^1) \to H^1_G(\text{Map}(X, S^1)) = \text{Ker} i^!$ is an isomorphism, but $H^2_G(S^1) \cong \text{Pic}_G(\ast) \cong \text{Hom}(G, S^1)$, and under the isomorphism $c^*$ corresponds to $c^!$, so Theorem 2 is proved.

Notice that $\text{Pic}_E(Y) \cong H^2(Y; Z)$ under the isomorphism which assigns to a line bundle $\lambda$ its first Chern class $c_1(\lambda)$.

**Corollary 3.** Let $f: X \to Y$ be a $G$-map, $X$ connected, $H^1(X; Z) = 0$, $s$ a $G$-line bundle over $X$, $t$ a $G$-line bundle over $Y$. Suppose $f^* c_1(i't) = c_1(i's)$, then there exists a homomorphism $\chi: G \to S^1$ such that $f^! i' = \chi s$.

**Proof.** Contemplate $s^{-1} \cdot f'i$. We have

$$c_1(i'(s^{-1} \cdot f'i)) = -c_1(i's) + c_1(i'f'i) = -c_1(i's) + f^* c_1(i't) = 0,$$

so $s^{-1} \cdot f'i$ is in the kernel of $i'$, hence in the image of $c'$—there exists a linear character $\chi: G \to S^1$ with $c' \chi = \chi \cdot 1 = s^{-1} \cdot f'i$, or $f'i = \chi s$, as claimed.
3. The strategy of proof. Let \( CP^{n-1} \) be a complex projective \((n-1)\)-dimensional space, \( s: S^{2n-1} \to CP^{n-1} \) the Hopf bundle over \( CP^{n-1} \). If \( \gamma: G \to U = U(n) \) is a representation, let \( s: S^{2n} \to CP^{n-1} \) be the Hopf bundle over \( CP^{n-1} \), hence an element in \( K_G(CP^{n-1}, \gamma) \) which we still call \( s \). Let \( R(G) = K_G(*) \) be the complex representation ring of \( G \), then \( \gamma \) is a free \( R(G) \)-module with \( 1, \ldots, s^n \) as basis and

\[
s^n - \gamma s^{n-1} + (\Lambda^2 \gamma)s^{n-2} - \cdots + (-1)^n \Lambda^n \gamma = 0,
\]

where \( \Lambda^i \gamma \) denotes the \( i \)th exterior power of \( \gamma \).

**Proposition 4.** Let \( \varphi: K_G(CP^{n-1}, \beta) \to K_G(CP^{n-1}, \gamma) \) be a homomorphism of \( R(G) \)-algebras with \( \varphi(s(\beta)) = \chi s(\alpha) \) for some linear character \( \chi: G \to S^1 \). Then \( \beta \) is similar to \( \chi \alpha \).

**Proof.** Let \( s(\alpha) = s, s(\beta) = t \). Then \( t \) satisfies

\[
t^n - \beta t^{n-1} + \cdots + (-1)^n \Lambda^n \beta = 0.
\]

Hence applying \( \varphi \) we have

\[
\chi^n s^n - \beta \chi^{n-1} s^{n-1} + \cdots + (-1)^n \Lambda^n \beta = 0,
\]

and multiplying with \( \chi^{-n} \) we obtain

\[
s^n - \beta \chi^{-1} s^{n-1} + \cdots + (-1)^n \Lambda^n (\beta \chi^{-1}) = 0.
\]

But

\[
s^n - \alpha s^{n-1} + \cdots + (-1)^n \Lambda^n \alpha = 0
\]

and \( K_G(CP^{n-1}, \alpha) \) is \( R(G) \)-free on \( 1, \ldots, s^{n-1} \). Comparing the coefficients of \( s^{n-1} \) we obtain \( \beta \chi^{-1} = \alpha \) in \( R(G) \) as claimed.

We shall now show how homotopy rigidity of linear actions on \( CP^{n-1} \) follows (compare \[11\], \[12\]). Let \( f: (CP^{n-1}, \alpha) \to (CP^{n-1}, \beta) \) be a \( G \)-map so that \( f^*: H^*(CP^{n-1}; Z) \to H^*(CP^{n-1}; Z) \) is an isomorphism. Let \( u = c_1(s) = c_1(i^*s(\alpha)) \), the first Chern class of the Hopf bundle \( s \). Then \( f^*u = u \) or \( -u \) since \( f^* \) is an isomorphism and \( H^2(CP^{n-1}; Z) \) is generated by \( u \). If \( f^*u = -u \), we replace \( f \) by \( c \circ f \) and \( \beta \) by \( \overline{\beta} \) (where \( c: CP^{n-1} \to CP^{n-1} \) is induced by conjugation in \( U = U(n) \)), so we may assume \( f^*u = u \), that is \( f^*c_1(i^!) = c_1(i^*s) \). We apply Corollary 3: there exists a linear character \( \chi: G \to S^1 \) such that \( f^* = \chi^s \). Applying Proposition 4 to \( \varphi = f^* \) we obtain that \( \beta \) is similar to \( \chi \alpha \). Recalling that we may have had to replace our original \( \beta \) by \( \overline{\beta} \) to obtain \( f^*u = u \) we obtain the homotopy rigidity result for linear actions on \( CP^{n-1} \).

We build our approach to linear actions on \( U/H \) on this special case of \( CP^{n-1} \). Suppose \( H \) is a friendly subgroup of \( U = U(n) \); there exists a nonzero vector \( v \in C^n \) such that \( hv = \lambda(h)v \) for all \( h \in H \) for some linear character \( \lambda \). We define a map \( \pi: U/H \to CP^{n-1} \) by \( \pi(uH) = [uv] \). If \( \alpha: G \to U \) is a representation then \( \pi \) is a \( G \)-map \( \pi_\alpha: (U/H, \alpha) \to (CP^{n-1}, \alpha) \).

**Proposition 5.** If \( H \) is a friendly subgroup of \( U = U(n) \) and \( \pi_\alpha \) is as above, then \( \pi_\alpha: K_G(CP^{n-1}, \alpha) \to K_G(U/H, \alpha) \) is a monomorphism.
PROOF. We may as well assume $H = U(n_1) \times U(n_2) \times \cdots \times U(n_k)$ with $n_k = 1$ and $v = e_n$, the last vector in the standard basis of $C^n$, then $\pi$ is induced by the inclusion $H \subset U(n-1) \times U(1)$. Let $T = U(1) \times \cdots \times U(1)$ be the standard $n$-torus of $U$ consisting of diagonal matrices, then $T \subset H \subset U$ induces a commutative diagram of projections

\[
(U/T, \alpha) \rightarrow (U/H, \alpha) \rightarrow (C^p, \alpha)
\]

and since $\rho^t$ is a monomorphism (see [18]), so is $\pi_\alpha^t$.

Now let $\alpha, \beta: G \rightarrow U$ be representations, $s = s(\alpha), t = s(\beta)$ the $G$-Hopf bundles on $(C^{p-1}, \alpha)$ and $(C^{p-1}, \beta)$, respectively. Let $f: (U/H, \alpha) \rightarrow (U/H, \beta)$ be a $G$-map such that $f: U/H \rightarrow U/H$ is a homotopy equivalence. Let $u = c_t(i^{*}_\beta s) = c_t(i^{*}_\beta k_t)$. If $f^s u = u$, then as before Corollary 3 says that there exists a linear character $\chi: G \rightarrow S^1$ such that $f^{*}_\beta s = s = \pi_\alpha^t(\chi s)$. Thus $f^t$ maps the image of $\pi_\beta$ into the image of $\pi_\alpha$. Since $\pi_\alpha$ is a monomorphism, we may define

\[
\phi = (\pi_\alpha^t)^{-1} f^{*}_\beta: K_G(C^{p-1}, \beta) \rightarrow K_G(C^{p-1}, \alpha)
\]

which, of course, is a map of $R(G)$-algebras and $\phi(t) = \chi t$, so Proposition 4 says that $\beta$ is similar to $\chi \alpha$. The catch, of course, is that there is no reason to expect that $f^* u$ is equal to $u$, so we have to do more work.

The group of $U$-maps $\text{Map}_U(U/H, U/H)$ is isomorphic to $N_U(H)/H$, where $N_U(H) = \{a \in U|aHa^{-1} = H\}$ is the normalizer of $H$ in $U$ (see Bredon [8], Samelson [17]). If $\gamma: G \rightarrow U$ is a representation and $k: U/H \rightarrow U/H$ is a $U$-map, then $k: (U/H, \gamma) \rightarrow (U/H, \gamma)$ is a $G$-map. Let $c: U \rightarrow U$ be given by $c(u) = \bar{u}$, the matrix with complex conjugate entries. We have chosen $H$ in its conjugacy class so that $c(H) = H$, hence $c: (U/H, \gamma) \rightarrow (U/H, \gamma)$ is a $G$-map. We have a homomorphism

\[
\psi: N_U(H)/H \times Z/2Z \rightarrow \text{Aut}(H^*(U/H; Z))
\]

given by $\psi(k, t) = k^* \circ c^t$. Stated in another way: the homomorphism $\psi$ defines an action of $N_U(H)/H \times Z/2Z$ on $H^*(U/H; Z)$. Of course the group $\text{Homeq}(U/H)$ of all homotopy classes of homotopy equivalences of $U/H$ also acts on $H^*(U/H; Z)$ by taking induced homomorphisms in cohomology. We now state several related conjectures.

**Conjecture B.** Let $H$ be a friendly subgroup of $U = U(n)$, $\pi: U/H \rightarrow C^{n-1}$ the standard map, $u = \pi^* c_1(s)$, where $s$ is the Hopf bundle on $C^{n-1}$, then the orbit of $u$ under $N_U(H)/H \times Z/2Z$ is the same as the orbit of $u$ under $\text{Homeq}(U/H)$.

**Proposition 6.** Conjecture B implies Theorem 1 (homotopy rigidity of linear actions on $U/H$).

**Proof.** We keep the notation of our earlier discussion: let $\alpha, \beta: G \rightarrow U = U(n)$ be representations, $f: (U/H, \alpha) \rightarrow (U/H, \beta)$ a $G$-map such that $f$:...
$U/H \to U/H$ is a homotopy equivalence, $u = \pi^*c_i(s)$. According to Conjecture B there exists an element $k$ of $N_U(H)/H \times \mathbb{Z}/2\mathbb{Z}$ such that $k^*f^*u = u$. Replace $f$ by $f \circ k$ (here we may have to replace $\alpha$ by $\al' \alpha$ if conjugation is involved). Then $f^*u = u$, so $i_f^*\pi^*_\beta = i^*_\alpha s$, and since $U/H$ is connected and simply connected, we obtain from Corollary 3 a linear character $\chi: G \to S^1$ such that $f^*\pi^*_\beta = \chi\pi^*_\alpha(s)$. So now letting $\varphi = \pi^*-1 f^*\pi^*_\beta$ we can apply Proposition 4 to conclude that $\beta$ is similar to $\chi\alpha$.

We shall prove an even stronger result for a multitude of subgroups of $U$:

**Conjecture C.** The map $\psi$ is an isomorphism of $N_U(H)/H \times \mathbb{Z}/2\mathbb{Z}$ onto the group of all algebra isomorphisms of $H^*(U/H; Z)$ if $H$ is a friendly subgroup of $U = U(n)$ and $n > 3$.

Notice that $U(2)/T^2 \cong S^2$, and in this case $\psi$ has a cyclic group or order 2 as a kernel. If $n > 3$, $\psi$ is a monomorphism.

Let us boldly walk even further on the limb: the following algebraic conjecture implies Conjecture B (and Conjecture C in a lot of cases).

**Conjecture D.** Let $T$ be the standard torus of $U = U(n)$, $\{e_1, e_2, \ldots, e_n\}$ the standard basis for $C^n$, let $\pi_i: U/T \to CP^n$ for $i = 1, \ldots, n$ be given by $\pi_i(uT) = [ue_i]$, $s$ the Hopf bundle on $CP^n$, let $x_i = \pi^*_i c_i(s)$. If $x \in H^2(U/T; Z)$ and $x^n = 0$ then there exists an integer $a$ and an $i$ in $\{1, 2, \ldots, n\}$ such that $x = ax_i$.

The algebraic data are easy to state: $H^*(U/T; Z) = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ modulo the ideal $I_n = (h_2, \ldots, h_n)$, where $h_i$ is the sum of all monomials of degree $i$ in $x_1, \ldots, x_n$ (see Borel [6])—for example, for $n = 4$, $h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$. It is important to notice that $n - 1$ appears above, not $n$—indeed $x_n = -x_1 - x_2 - \cdots - x_{n-1}$. The group $N_U(T)/T$ is $S_n$, the symmetric group on $n$ letters which acts on $H^*(U/T; Z)$ by permuting the $x_1, \ldots, x_n$. Conjecture D is trivial to prove for $n = 3$. For $n = 4$, the algebra is already delightfully complicated and a hint is helpful: examine the solutions of $x^4 = 0$ first over $\mathbb{Z}/3\mathbb{Z}$ and then exploit the fact that multiplication by $x_1$ from $H^6(U(4)/T^4, \mathbb{Z}/3\mathbb{Z})$ to $H^8(U(4)/T^4, \mathbb{Z}/3\mathbb{Z})$ has kernel of dimension one to show that if $x_1 + y$ is a solution of $x^4 = 0$ over $\mathbb{Z}$ and $y = bx_2 + cx_3$ then for all natural numbers $k$ we have $3^k|y$ implies $3^k+1|y$, so $y = 0$.

The limb is beginning to creak ominously, but let's take one more step:

**Conjecture E.** If $H$ is a connected subgroup of maximal rank of $U = U(n) \text{ and } \text{Homeq}(U/H)$ is the group of homotopy classes of homotopy equivalences of $U/H$ then $N_U(H)/H \times \mathbb{Z}/2\mathbb{Z}$ is a normal subgroup of $\text{Homeq}(U/H)$ if $n > 3$.

One reason for thinking wishfully about Conjecture E is that it would give a beautifully simple proof of Conjecture C for $H = T$, the maximal torus of $U(n)$ and $\text{Homeq}(U/H)$ is the group of homotopy classes of homotopy equivalences of $U/H$ then $N_U(H)/H \times \mathbb{Z}/2\mathbb{Z}$ is a normal subgroup of $\text{Homeq}(U/H)$ if $n > 3$.

**4. Algebra automorphisms of $H^*(U/H; Z)$.** We shall prove Conjecture C for $U = U(n)$, $H = U(n - k) \times T^k$, $n > \max\{2k, k + 2\}$. As before, let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $C^n$ and let $\pi_i: U/H \to CP^n$ be the projection $\pi_i(uH) = [ue_{n-k+i}]$ for $i = 1, \ldots, k$. Let $y \in H^2(CP^{n-1}; Z)$ be the Chern class of the Hopf bundle and let $x_i = \pi^*_i(y)$; then $H^*(U/H);$
$Z = Z[x_1, \ldots, x_k]/I$, where the ideal $I = (h_{n-k+1}, \ldots, h_n)$ and $h_j$ is the sum of all monomials of degree $j$ in $x_1, \ldots, x_k$. A free basis for $H^*$ is given by $x^E = x_1^{e_1}x_2^{e_2}\ldots x_k^{e_k}$, where $0 \leq e_i < n - k + i$ (see Borel [6]). The group $N_U(H)/H$ is $S_k$, the symmetric group on $k$ letters, and it acts on $H^*(U/H; Z)$ by permuting $x_1, \ldots, x_k$. We examine the case of $k = 2$ more closely.

**Lemma 7.** If $u = ax_1 + bx_2$ is an element in $H^2(U(m+2)/U(m) \times T^2; Z)$ with $u^{2m} = 0$, then either $a = 0$ or $b = 0$.

**Proof.** We first claim that if both $a$ and $b$ are nonzero and $u^{2m+1} = 0$ then $a = b$. Notice that $x_1^{m+2} = x_2^{m+2} = 0$ (since both come from $CP^{m+1}$) and $H^{4m+2}$ has $x_1^mx_2^{m+1}$ as basis. Moreover, $x_1^{m+1}x_2^m = -x_1^mx_2^{m+1}$. We have

$$0 = u^{2m+1} = (ax_1 + bx_2)^{2m+1} = \left(\frac{2m+1}{m+1}\right)a^{m+1}b^{m+1}x_1^{m+1}x_2^m + \left(\frac{2m+1}{m}\right)a^mb^{m+1}x_1^mx_2^{m+1},$$

so $a \neq 0, b \neq 0$ implies $a = b$. If now, in addition, $u^{2m} = 0$, then we have

$$0 = \left(\frac{2m}{m+1}\right)a^{2m}x_1^mx_2^{m-1} + \left(\frac{2m}{m}\right)a^mx_1^mx_2^m$$

$$+ \left(\frac{2m}{m-1}\right)a^mx_1^m-x_2^{m+1},$$

but $H^{4m}$ has $\{x_1^mx_2^m, x_1^{m-1}x_2^{m+1}\}$ as basis and

$$x_1^{m+1}x_2^{m-1} = -x_1^mx_2^m - x_1^{m-1}x_2^{m+1},$$

so the above sum reduces to

$$0 = \left\{\left(\frac{2m}{m}\right) - \left(\frac{2m}{m+1}\right)\right\}a^mx_1^mx_2^m,$$

so since $\binom{2m}{m} \neq \binom{2m+1}{m+1}$ for all $m$ it follows that $a = 0$, a contradiction to our temporary hypothesis that $a \neq 0$ and $b \neq 0$.

**Corollary 8.** Let $v \in H^2(U(m+k)/U(m) \times T^k; Z)$ be an element such that $v^{m+k} = 0$. If $m \geq k$ then $v = ax_i$ for some $i$ in $\{1, \ldots, k\}$.

**Proof.** By applying a suitable element of $S_k$ we can assume that the coefficient of $x_1$ is nonzero. We wish to show that the coefficient of $x_i$ for $i \neq 1$ is zero—and, of course, it is sufficient to prove this for $i = 2$. Consider the standard map

$$j: U(m+2)/U(m) \times T^2 \to U(m+k)/U(m) \times T^k$$

induced by the standard inclusion $C^{m+2} \subset C^{m+k}$ under which $j^*x_1 = x_i$, $j^*x_2 = x_2$, $j^*x_j = 0$ for $i > 2$. Inspect $u = j^*v$; then $u^{m+k} = 0, m + k < 2m$; hence $a \neq 0$ implies that the coefficient of $x_2$ is zero.

We are now ready to prove Conjecture C for $U(m + k)/U(m) \times T^k$.

**Theorem 9.** If $n > \max\{2k, k+2\}$, $U = U(n), H = U(n-k) \times T$, then the map $\psi$ is an isomorphism of $N_U(H)/H \times Z/2Z$ onto the group of all algebra isomorphisms of $H^*(U/H; Z)$.

**Proof.** We first prove that $\psi$ is onto. Let $\varphi: H^*(U/H; Z) \to H^*(U/H; Z)$
be an algebra automorphism; then \( q(x_1) = u \) is an element with \( u^n = 0 \) (this since \( x_1^n = 0 \)) and because \( n > 2k \), Corollary 8 is applicable, so \( u = ax \) for some \( i \) and \( a = 1 \) or \(-1\). By using elements of \( N_u(H)/H = S_k \) we can normalize \( \varphi \) (using \( c^* \) if necessary) to have \( \varphi(x_1) = x_1 \). We claim \( \varphi \) is the identity. If not, use \( S_k \) to arrange \( \varphi(x_2) = -x_2 \). Now consider \( \varphi \) as an automorphism of \( Z[x_1, \ldots, x_k] \) (remember: there are no relations among the generators in \( H^2 \)). The relations in grading \( 2n - 2k + 2 \) are generated by \( h_{n-k+1} \) so we must have \( \varphi h_{n-k+1} = \pm h_{n-k+1} \), but \( \varphi(x_1^{n-k+1}) = x_1^{n-k+1} \) and \( \varphi(x_1^{n-k}x_2) = -x_1^{n-k}x_2 \), so \( \varphi(x_2) = -x_2 \) is impossible, and we have shown that \( \psi \) is onto.

To prove that \( \psi \) is one-to-one is even easier: since \( m > 2 \) each \( \sigma \in N_u(H)/H = S_k \) maps \( x_i \) into some \( x_i \), \( c^*x_1 = -x_1 \), so the kernel of \( \psi \) is contained in \( S_k \times 0 \), but we have already noticed that \( \psi|S_k \times 0 \) is faithful, so \( \psi \) is one-to-one, as claimed.

**BIBLIOGRAPHY**


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