for a beginner. This book is an important and valuable addition to a growing secondary literature, which is perhaps best absorbed by wandering back and forth from one book to another and thence to the primary sources as quickly as possible.

We wish to add our thanks and compliments to J. N. Crossley for translating this work into (very readable) English.

REFERENCES


SOLOMON FEFERMAN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 1, January 1979
©American Mathematical Society 1979
0002-9904/79/0000-0010/$01.75


In this monograph the author gives a highly readable introduction to two topics, namely the Kalman filter and the stochastic linear regulator problem. These two topics have been called the "bread and butter" of modern stochastic systems theory. They have the fortunate feature that the mathematical techniques used are elegant, while at the same time the results have been quite widely used in engineering and other applications. Rather modest background is needed to read the book. The equivalent of introductory real-analysis, probability, and some familiarity with elementary linear systems theory should suffice.

The linear estimation problem is as follows. Given random variables \( X, Y_s \) for \( s \in S \), all of zero mean and finite variance, find the approximation \( \hat{X} \) to \( X \).
in the Hilbert space spanned by \( \{ Y_s \} \) which is best in mean square. One may think of \( X \) as being unobservable, and \( \{ Y_s \} \) as representing the available data. Often one has not a single random variable \( X \), but rather a stochastic process \( X_t \). Moreover, new data is arriving continually; the observations at time \( t \) are \( \{ Y_s \} \) for \( s < t \), with now the index set \( S = S_t \) the interval \([0, t]\). What is needed is a computationally tractable scheme for updating the best linear estimate \( \hat{X}_t \). In 1960 Kalman introduced such a technique, which was then developed further by Kalman-Bucy and others. It now goes by the name “Kalman filter”.

In the Kalman filter model \( X_t, Y_t \) obey linear differential equations

\[
\begin{align*}
\dot{X}_t &= A_t X_t \, dt + C_t \, dV_t, \\
\dot{Y}_t &= H_t X_t \, dt + G_t \, dW_t,
\end{align*}
\]

where \( A_t, C_t, H_t, G_t \) are nonrandom, and \( V_t, W_t \) are certain stochastic processes with orthogonal increments. [The stochastic processes \( X_t, Y_t, V_t, W_t \) are allowed to be vector-valued; then \( A_t, C_t, \ldots \) are matrices.] The main Kalman filter results are: (a) \( \hat{X}_t \) also obeys a certain linear differential equation, driven by the observation process \( Y_t \); (b) the covariance matrix of the error \( X_t - \hat{X}_t \) does not depend on the observations. In fact, these covariance matrices can be precomputed by solving a system of ordinary differential equations of Riccati type. A key to the method is to introduce the “innovations process”

\[
V_t = Y_t - \int_0^t H_s \hat{X}_s \, ds.
\]

This process has orthogonal increments; and \( \{ V_s \} \) for \( s < t \) generate the same Hilbert space as \( \{ Y_s \} \) for \( s < t \).

In the stochastic linear regulator problem, equation (1) is replaced by

\[
\dot{X}_t = (A_t X_t + B_t U_t) \, dt + C_t \, dV_t,
\]

where \( U_t \) represents a control used at time \( t \). The control may depend on past observations \( \{ Y_s \} \) for \( s < t \). The control process \( U_t \) is to be chosen to minimize a quadratic of the form

\[
E \left\{ \int_0^T \left( X_t' Q_t X_t + U_t' R_t U_t \right) dt + X_T' F X_T \right\}.
\]

A “separation principle” reduces this problem to two others. One is a Kalman filter to find \( \hat{X}_T \), the other is a deterministic linear regulator problem in which the estimate \( \hat{X}_t \) is regarded as the true state of the system. A solution to the deterministic linear regulator problem can be found by solving a system of ordinary differential equations of Riccati type.


WENDELL H. FLEMING