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CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}(2, \mathbb{C})$

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Let $\mathfrak{g}$ be a nonabelian Lie algebra over an algebraically closed field $K$ of characteristic 0. One is interested in the (algebraically) irreducible representations of $\mathfrak{g}$ acting on a vector space which is allowed to be infinite dimensional. The subject of enveloping algebras is largely concerned with these, but even in the simplest nonabelian case, with $\mathfrak{g} = \mathfrak{h}$ the 3-dimensional (nilpotent) Heisenberg algebra, as Dixmier remarks in discussing the situation when $K = \mathbb{C}$ in the preface to [2], "a deeper study reveals the existence of an enormous number of irreducible representations of $\mathfrak{h} \ldots$ It seems that these representations defy classification. A similar phenomenon exists for $\mathfrak{g} = \mathfrak{sl}(2)$, and most certainly for all noncommutative Lie algebras."

However, as we shall see, the situation for $\mathfrak{h}$ and for $\mathfrak{sl}(2)$ turns out to be far nicer than hoped for. Indeed we announce here a determination and classification of all irreducible representations of $\mathfrak{h}$, of $\mathfrak{sl}(2)$, and of the 2-dimensional nonabelian Lie algebra, and thus of the prototypes respectively of nilpotent, simple, and solvable Lie algebras. As a guide to the meaning of "classification" and because our results use the same invariants, consider a classical situation of an (associative) algebra for which the irreducible representations have long been classified, namely, the algebra $B$ of formal linear differential operators with rational function coefficients, i.e., $B = K(q)[p]$, the (noncommutative) polynomials in an indeterminate $p$ where multiplication is determined by the relation $pq - qp = 1$. Then $B$ is a left principal ideal domain. Therefore [3] a $B$-module $M$ is simple if and only if $M \cong B/Bb$ for some $b \in B$ which is irreducible (i.e., $b = ac$ implies $a$ or $c$ is a unit); and $B/Bb \cong B/Ba$ if and only if $a$ and $b$ are similar, i.e., there exists $c \in B$ such that $(b, c) = 1$ and $a = [b, c]c^{-1}$ where $(b, c)$ is a...
right g.c.d. and \([b, c]\) a left l.c.m. (which always exist) (similar is the noncommutative generalization of associate).

The subalgebra \(K[q][p]\) of \(B\) generated by \(p, q\) is the Weyl algebra \(A_1\). Since \(A_1 \cong U_h/U_b(z - \alpha)\) for \(0 \neq z \in \text{center } \mathfrak{h}\) and \(0 \neq \alpha \in K\), the problems of finding the irreducible representations for \(A_1\) and for \(\mathfrak{h}\) are equivalent. Our solution for this problem as well as for \(\mathfrak{so}(2)\) involves the new notion of preserving, defined in terms of certain polynomials which we now introduce. For \(\alpha \in K\) let \(\mu_\alpha\) denote the valuation of \(K(q)\) determined by the prime \(q - \alpha\) of \(K[q]\), and extend \(\mu_\alpha\) to a function (also denoted \(\mu_\alpha\)) on \(B\) by setting \(\mu_\alpha(\Sigma_f b_f(q)p^j) = \min \{\mu_\alpha(b_f(q)) - jy \geq 0\}\). Then define \(\theta_{\alpha,b}(\lambda) \in K[\lambda] (\alpha \in K, b = \Sigma_f b_f(q)p^j \in B)\) by

\[
\theta_{\alpha,b}(\lambda) = \sum_f \left\{(q - \alpha)^{-\mu_{\alpha b} - f} \theta_f(\alpha)\right\}\left(-1\right)^{j}(\lambda + 1) \cdots (\lambda + j - 1).
\]

(It can be proved that \(\mu_\alpha\) is a valuation on \(B\), and extends to a valuation on the quotient division ring whose residue field is \(K(\lambda)\); then with \(\varphi_\alpha\) the corresponding place, \(\theta_{\alpha,b}(\lambda) = \varphi_\alpha((q - \alpha)^{-\mu_{\alpha b} - b})\).) Call \(b\) \(\alpha\)-preserving if \(\theta_{\alpha,b}(\lambda)\) has no positive integral root, and preserving it is \(\alpha\)-preserving for all \(\alpha \in K\). It can be shown that \(b\) is preserving if it is \(\alpha\)-preserving for a certain finite set of \(\alpha\)'s, in particular (when \(b\) is normalized to be in \(A_1\)) for the set of roots of the leading coefficient \(b_r(q)\); thus if \(K = \mathbb{C}\) the property of \(b\) being preserving is computable given the roots of \(b_r(q)\).

The \(A_1\)-module \((K[p], q - \alpha\) acts as \(-d/dp)\) is simple and is precisely the simple \(A_1\)-module for which \(q\) has \(\alpha\) as an eigenvalue.

**Theorem 1.** If \(a \in B\) is irreducible and preserving then the \(A_1\)-module \(A_1/A_1 \cap Ba\) is simple.

**Theorem 2.** If \(M\) is a simple \(A_1\)-module then either \(M \cong (K[p], q - \alpha\) acts as \(-d/dp)\) for some \(\alpha \in K\) or \(M \cong A_1/A_1 \cap Ba\) for some \(a\) as in Theorem 1.

Since the \(A_1/A_1 \cap Ba\) above have no eigenvector for \(q\), the following completes the classification of the simple \(A_1\)-modules.

**Theorem 3.** Two simple \(A_1\)-modules \(A_1/A_1 \cap Ba, A_1/A_1 \cap Bb\) are isomorphic if and only if \(a\) and \(b\) are similar (in \(B\)).

Now consider the case of \(\mathfrak{g} = \mathfrak{sl}(2, K) = \mathfrak{sl}\), with canonical basis \(e, f, h\). For \(\beta \in K\) the map \(e \rightarrow q, h \rightarrow 2qp - \beta, f \rightarrow -(qp - \beta)p\) extends to a homomorphism \(\rho_\beta\) of \(U\mathfrak{g}\) to \(B\). The simple \(\mathfrak{g}\)-modules for which \(e\) has an eigenvector \(v\) (with eigenvalue \(\alpha\)) are as follows: if \(\alpha = 0\), the highest weight modules \(L(\beta)\) (\(\beta \in K\)) (with \(hv = \beta v\); if \(\alpha \neq 0\), the simple Whittaker module \(W_{h\beta}^\alpha(\alpha)\) (see \([1], [4]\); \(\alpha = \eta(e)\)), with basis \(v^0 = v, t^1, \ldots\) where \(ht^2 = 2t^{d+1}, et^i = \alpha(t - 1)^i, ft^i = \alpha^{-1}(t + 1)^i(-t^2 + (\beta^2 + 2\beta)/4)\). The only isomorphisms among these
are $\text{Wh}_\beta(\alpha) \cong \text{Wh}_\delta(\alpha)$ whenever $\beta^2 + 2\beta = \delta^2 + 2\delta$. For any $\beta \in K$ write $\beta'$ for the other root of $\lambda^2 + 2\lambda = \beta^2 + 2\beta$ i.e., $\beta' = -\beta - 2$.

**Theorem 4.** Suppose $a \in U\mathfrak{g}$, $\beta \in K$, $\rho_\beta a$ is irreducible (in $B$) and $\rho_\beta a$ and $\rho_\beta' a$ are preserving. Then the $U\mathfrak{g}$-module $\rho_\beta U\mathfrak{g} / \rho_\beta U\mathfrak{g} \cap B(\rho_\beta a)$ is simple.

**Theorem 5.** If $M$ is a simple $U\mathfrak{g}$-module then either $M \cong L(\beta)$ for some $\beta \in K$ or $M \cong \text{Wh}_\beta(\alpha)$ for some $\alpha$, $\beta \in K$, $\alpha \neq 0$, or $M \cong \rho_\beta U\mathfrak{g} / \rho_\beta U\mathfrak{g} \cap B(\rho_\beta a)$ for some $a$ as in Theorem 4.

Again the following completes the classification.

**Theorem 6.** Two simple $U\mathfrak{g}$-modules $\rho_\beta U\mathfrak{g} / \rho_\beta U\mathfrak{g} \cap B(\rho_\beta a)$, $\rho_\beta U\mathfrak{g} / \rho_\beta U\mathfrak{g} \cap B(\rho_\beta b)$ are isomorphic if and only if $\beta^2 + 2\beta = \delta^2 + 2\delta$ and $\rho_\beta a$ and $\rho_\beta b$ are similar (in $B$).

Analogous results hold for the 2-dimensional nonabelian Lie algebra, realized say as the subalgebra $\mathfrak{b} = K\mathfrak{h} + Ke$ of $\mathfrak{g}$, with the following changes: the simple $\mathfrak{b}$-modules for which $e$ has an eigenvector are $\text{Wh}_\beta(\alpha)$ (for $\alpha \neq 0$) and, for each $\delta \in K$, $Kv \subseteq L(\delta)$; restrict $\beta$ to 0 and change the condition on preserving to the condition that $\rho_\beta a$ be preserving and $\theta_{\rho_\beta a}(\lambda) \in K$ (or equivalently, $a = e^u( ec + \alpha)$ for some $u \in \mathbb{N}$, $c \in U\mathfrak{b}$ and $0 \neq \alpha \in K$).

The ring $B$ is the localization of its subrings $A_1$ and $\rho_\beta U\mathfrak{g}$ with respect to the multiplicative subset $S = K[q] - \{0\}$.

**Theorem 7.** Every simple $B$-module $N$ contains a unique simple $A_1$-submodule $\psi N$ and, for every $\beta \in K$, a unique simple $\rho_\beta U\mathfrak{g}$ submodule $\psi_\beta N$; $\psi N$ (resp. $\psi_\beta N$) is contained in every nonzero $A_1$- (resp. $\rho_\beta U\mathfrak{g}$-) submodule of $N$. Also $B N \cong S^{-1}(\psi N) \cong S^{-1}(\psi_\beta N)$, and if $M$ is a simple $S$-torsionfree $A_1$- (resp. $\rho_\beta U\mathfrak{g}$-) module then $\psi(S^{-1}M)$ (resp. $\psi_\beta(S^{-1}M)$) $\cong M$. Thus the map $N \rightarrow \psi N$ (resp. $N \rightarrow \psi_\beta N$) sets up a bijection between the set of isomorphism classes of simple $B$-modules and the set of isomorphism classes of $S$-torsionfree simple $A_1$-modules (resp. $\rho_\beta U\mathfrak{g}$-modules with the Casimir element $4fe + h^2 + 2h$ acting as $\beta^2 + 2\beta$).

Here is a formula involving the $\theta_{\alpha, \beta}(\lambda)$ which helps to explain their relevance to modules. If $a \in A$, then for the action on $(K[p], q - \alpha$ acts as $-d/dp$), for every positive integer $s$ we have

\begin{equation}
(q - \alpha)^{-s} \alpha^a \cdot p^{s-1} = \theta_{\alpha, \beta}(s)p^{s-1} + \text{lower terms}.
\end{equation}

Somewhat similar formulas hold for the actions of $U\mathfrak{g}$ on $\text{Wh}_\gamma(\alpha)$ and $L(\delta)$. The proof of Theorem 1 begins by showing that a maximal ideal $J$ of $A$ properly containing $A \cap Ba$ intersects $S$, and so $q$ has an eigenvector on $A/J$. Then one uses $\alpha$-preserving and (1). Theorem 4 is similar. The remaining theorems use properties of minimal annihilators and localizations. Theorems 2, 5 and 7 also depend on the following.
Lemma. If \( b \in B \), there exists \( d \in S \) such that \( bd^{-1} \) is preserving.

The proof of Theorem 7 also uses Theorem 1 and 4; if \( N = B/Bb \) where \( b \) is preserving then \( \psi N = (A_1 + Bb)/Bb \).

REFERENCES


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