MAXIMAL FUNCTIONS:
A PROOF OF A CONJECTURE OF A. ZYGMUND

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In \( \mathbb{R}^n \) let us consider the family \( B_n \) of parallelepipeds with sides parallel to the coordinate axes. We may ask for conditions upon the locally integrable function \( f \) in order that

\[
\lim_{x \in \mathbb{R} \in B_n} \frac{1}{\mu(R)} \int_{R} f(y) \, d\mu(y) = f(x)
\]

\[\text{diam}(R) \to 0\]  

a.e. \( x \), where \( \mu = \text{Lebesgue measure in } \mathbb{R}^n \).

In 1935 B. Jessen, J. Marcinkiewicz and A. Zygmund [1] showed that \([*]\) holds so long as \( f \in L(1 + (\log^+ L)^{n-1})(\mathbb{R}^n) \) locally. Furthermore this result is the best possible in the following sense: if \( \psi(t) \) is an Orlicz’s space defining function such that \( \psi(t) = o(t(\log t)^{n-1}) \), \( t \to \infty \), then statement \([*]\) is false for a typical \( L_{\psi} \)-function (typical in the sense of Baire’s category). Of course the case \( n = 1 \) was known before as Lebesgue’s Differentiation theorem.

The following natural problem was proposed by A. Zygmund: given a positive function \( \Phi \) on \( \mathbb{R}^2 \), monotonic on each variable separately, let us consider the differentiation basis \( B_\Phi \) in \( \mathbb{R}^3 \) defined by the two parameters family of parallelepipeds whose sides are parallel to the rectangular coordinate axis and whose dimensions are given by \( s \times t \times \Phi(s, t) \), \( s, t \) positive real numbers. For which locally integrable functions \( f \) is statement \([*]\) true with respect to the family \( B_\Phi \)?

In general the differentiation properties of \( B_\Phi \) must be, at least, not worse than \( B_3 \), the basis of all parallelepipeds in \( \mathbb{R}^3 \) whose sides have the direction of the coordinate axes, and, of course, not better than \( B_2 \). A. Zygmund conjectured after his 1935 paper that \( B_\Phi \) behaves like \( B_2 \). This conjecture is now a theorem with applications to a.e. convergence of Poisson Kernels associated to certain symmetric spaces.

THEOREM. (a) \( B_\Phi \) differentiates integrals of functions which are locally in \( L(1 + \log^+ L)(\mathbb{R}^3) \), that is
so long as \( f \) is locally in \( L(1 + \log^+ L)(\mathbb{R}^3) \), where \( \mu \) denotes Lebesgue measure in \( \mathbb{R}^3 \).

(b) The associated maximal function

\[
M_\Phi f(x) = \sup_{x \in R, R \in \mathcal{B}_\Phi} \frac{1}{\mu(R)} \int_R |f(y)| \, d\mu(y)
\]

satisfies the inequality

\[
\mu\{M_\Phi f(x) > \alpha > 0\} \leq C \int_{\mathbb{R}^3} \frac{|f(x)|}{\alpha} \left\{1 + \log^+ \frac{|f(x)|}{\alpha}\right\} \, d\mu(x)
\]

for some universal constant \( C < \infty \).

The proof is based on the following geometric argument:

**Covering Lemma.** Let \( \mathcal{B} \) be a family of dyadic parallelepipeds in \( \mathbb{R}^3 \) satisfying the following monotonicity property: if \( R_1, R_2 \in \mathcal{B} \) and the horizontal dimensions of \( R_1 \) are both strictly smaller than the corresponding dimensions of \( R_2 \), then the vertical dimension of \( R_1 \) must be not bigger than the vertical dimension of \( R_2 \).

Then the family \( \mathcal{B} \) satisfies the exponential type covering property, that is:

- (i) \( \mu\{\bigcup R_\alpha\} \leq C \mu\{\bigcup R_j\} \),
- (ii) \( \int_{\mathbb{R}^3} \exp(\Sigma x_{R_j}(x)) \, d\mu(x) \leq C \mu\{\bigcup R_j\} \)

for some universal constant \( C < \infty \).

**Application.** Consider

\[
\mathbb{R}^3 = \{X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}, \text{ real, symmetric, } 2 \times 2\text{-matrices}\},
\]

and the cone \( \Gamma = \{X \in \mathbb{R}^3, \text{ positive definite}\} \). Then \( T_\Gamma = \text{tube over } \Gamma = \text{Siegel's upper half-space} = \{X + iY, Y \text{ positive definite}\} \).

For each integrable function \( f \) in \( \mathbb{R}^3 \) we have the "Poisson integral",

\[
u(X + iY) = P_Y * f(x), \quad Y \in \Gamma,
\]

where

\[
P_Y(X) = C \frac{\det Y}{{\det(X + iY)}^3}
\]

and we may ask the following question: for which functions \( f \) is it true that \( u(X + iY) \to f(X) \), a.e. \( X \), where \( Y \to 0 \)?
It is a well-known fact that if $Y = cI = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \to 0$, then $u(X + iY) \to f(X)$, a.e. $X$ for integrable functions $f$. On the other hand if $Y \to 0$ without any restriction than a.e. convergence fails for every class $L^p(\mathbb{R}^3)$, $1 \leq p \leq \infty$.

Here we can settle the case

$$Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \to 0$$

because an easy computation shows that

$$Mf(X) = \text{Sup}_{Y} |u(X + iY)|$$

where

$$Y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$$

is majorized, in a suitable sense, by $M_\phi f$ with $\Phi(s, t) = (s \cdot t)^{1/6}$. Therefore we have convergence for $L(1 + \log^+ L)(\mathbb{R}^3)$ and, since $Mf \geq CM_\phi f$ is also true for some $c > 0$, $L(1 + \log^+ L)(\mathbb{R}^3)$ is the best class for which almost everywhere convergence holds.

REFERENCES

2. A. Córdoba, $s \times t \times \Phi(s, t)$, Institut Mittag-Leffler Report No. 9, 1978.

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