too can cartesian diagrams." The reader will also have difficulty in many places where letters which should be adorned with bars or tildes (thus, $\bar{H}$ or $\tilde{C}$) appear without their adornments, and thus look as if they mean something else.

This brief sample of small errors of typography and presentation are given to warn the reader that he will need to study the text very carefully, not only for its mathematical content. They are not intended to disparage the undoubted value of this book to all those concerned with this important branch of algebraic topology.

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Harmonic analysis on real reductive groups, by V. S. Varadarajan, Lecture Notes in Math., vol. 576, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 521 pp., $17.20.

The study of harmonic analysis on real semisimple Lie groups has proceeded in three major currents. One of these has been via the study of more general locally compact groups. There are, for example, general results about character theory, induced representations, and the Plancherel theorem. Good references for this work are the books of Dixmier and Mackey [3], [8].

In the other extreme, there have been studies of specific groups, most frequently $SL(2, R)$, the group of 2×2 real matrices with determinant one. Although this is the simplest possible example of a semisimple real Lie group, studies of $SL(2, R)$, two of the earliest being papers by E. Wigner in 1939 and V. Bargmann in 1947, provided invaluable inspiration for the general theory [2], [13].

Finally, there has been the study of real semisimple Lie groups in general. The pioneering work in this area is due almost entirely to Harish-Chandra. This work exploits the rich structure theory of semisimple groups and the connections between analysis on these groups and the abelian Fourier analysis on their Lie algebras and Cartan subgroups.

The building blocks of harmonic analysis are irreducible unitary representations. The set $\hat{G}$ of equivalence classes of irreducible (continuous) representations of a locally compact group G is called the unitary dual of G. For $\pi \in \hat{G}$ and $f \in L^1(G)$, the operator-valued Fourier transform of f is given by $\pi(f) = \int_G f(x)\pi(x)\,dx$, where dx is Haar measure on G. The scalar-valued Fourier transform is $\hat{f}(\pi) = \operatorname{trace} \pi(f)$, if $\pi(f)$ has a well-defined trace as an operator on the representation space.

A Plancherel measure for G is a positive measure on $\hat{G}$ such that for $f \in L^1(G) \cap L^2(G)$, $\|\pi(f)\|$ is finite for μ-almost all $\pi \in \hat{G}$, and $\int_G |f(x)|^2 dx = \int_{\hat{G}} \|\pi(f)\|^2 d\mu(\pi)$. Here $\|\cdot\|$ denotes the Hilbert-Schmidt norm. Plancherel's theorem says that for a large class of locally compact groups (including compact and semisimple Lie groups, see [3]), Plancherel measure exists on $\hat{G}$, and is unique once a Haar measure dx on G is fixed. An easy consequence of Plancherel's theorem is the Plancherel formula:

$$f(e) = \int_{\hat{G}} \hat{f}(\pi)\, d\mu(\pi), \qquad f \in C_c^\infty(G).$$

For compact Lie groups, all irreducible representations are finite dimensional, and for $(\pi, H) \in \hat{G}$, $\mu(\pi) =$ dimension H. Further, $\hat{f}(\pi) = \int_G f(x)\theta_\pi(x)\, dx$, where $\theta_\pi(x) =$ trace $\pi(x)$ is called the character of π. θ_π is a central function on G, an eigenfunction for left and right invariant differential operators on G, and determines the equivalence class of $\pi \in \hat{G}$.

In contrast, noncompact semisimple Lie groups have no finite-dimensional unitary representations aside from the trivial representation. Thus the unitary operators $\pi(x)$ cannot have finite traces. However, for $f \in C_c^\infty(G)$, $\pi(f)$ is of trace class and the distribution θ_π: $f \mapsto$ trace $\pi(f) = \hat{f}(\pi)$ is called the character of π. The distribution θ_π is called an invariant eigendistribution on G because it has properties analogous to those of the function θ_π in the compact case. One of the important theorems of Harish-Chandra is that any invariant eigendistribution on a semisimple real Lie group is given by integration against a function on G. That is, there is a locally integrable function θ_π on G so that $\hat{f}(\pi) = \int_G f(x)\theta_\pi(x)\, dx, f \in C_c^\infty(G)$.

The Plancherel measure for semisimple Lie groups is not discrete as it is for compact groups. An element $\pi \in \hat{G}$ is called a discrete series representation if it has strictly positive Plancherel measure. This positive number, $\mu(\pi)$, is called the formal degree of π. Thus the discrete series, denoted $\hat{G}_d$, is the subset of $\hat{G}$ on which Plancherel measure is discrete. Discrete series representations are also known as square-integrable representations because they can be characterized as the elements of $\hat{G}$ with L^2 matrix coefficients. Harish-Chandra proved that $\hat{G}_d$ is nonempty if and only if G has a compact subgroup of rank equal to that of G, and he gave an explicit construction of all discrete series characters.

The discrete series play a central role in harmonic analysis on real semisimple Lie groups. Given the discrete series for G and for certain lower-dimensional subgroups of G, μ-almost all elements of $\hat{G}$ can be constructed by the process of unitary induction. There is one technical problem in setting up this induction. If one starts with a connected, acceptable, real semisimple Lie group with finite center (as is customary), the subgroups from which representations will be induced need not satisfy any of these conditions. Thus it is necessary to develop the character theory for a larger class of groups, which was defined by Harish-Chandra, and is termed by Varadarajan the "reductive groups of class $\mathcal{H}$." This class contains the semisimple Lie groups satisfying the above conditions and has good hereditary properties, but does not include all reductive, or even all semisimple groups. Most of Harish-Chandra's theorems were proved first for semisimple groups, and later extended to the reductive groups.

Roughly speaking, reductive groups of class $\mathcal{H}$ are allowed to have a center and a finite number of connected components. An important example is $GL(2, R)$, the group of all invertible 2×2 real matrices. $GL(2, R)$ has as center the group R^x of scalar matrices. It has two components, those matrices with positive determinant, and those with negative determinant. Finally, its commutator subgroup is the semisimple group $SL(2, R)$.

Varadarajan's book, *Harmonic analysis on reductive groups*, gives a detailed exposition of the work of Harish-Chandra on invariant eigendistributions and the discrete series for reductive groups. It ends with the determination of the characters and formal degrees of the discrete series representations. It does not include any discussion of the induced representations which belong to the support of the Plancherel measure, the theory of Eisenstein integrals, or Harish-Chandra's complete version of the Plancherel theorem.

The analysis required to construct the discrete series characters is formidable. It is necessary to develop a Schwartz space of functions vanishing rapidly at infinity and study tempered distributions, those distributions which can be extended from $C_c^\infty(G)$ to this larger space. Many proofs require descent to the Lie algebra, so that all the machinery of invariant distributions and differential operators must be developed first for reductive Lie algebras. For example, the theorem that an analytic differential operator which kills all invariant C^∞ functions, kills all invariant distributions is first proved at the Lie algebra level by very subtle analysis. The extension to the group is then a fairly easy consequence.

This analytic machinery was all developed by Harish-Chandra before 1965. The complete theory of the discrete series for semisimple Lie groups appeared in his famous Acta papers of 1965–1966 **[4a]**. The formal details for the extension to the reductive groups of class $\mathcal{H}$ however were not available in print until 1975 **[4c]**.

Experts should find Varadarajan's book a convenient reference for material which is scattered among many of Harish-Chandra's papers. Varadarajan follows Harish-Chandra's point of view closely. However, some of Harish-Chandra's proofs, for example, the construction of the discrete series and the proof that the orbital integral of a Schwartz function is again a Schwartz function are simplified significantly. Also, Varadarajan's detailed introductions provide an overview of the development which is very difficult to glean from Harish-Chandra's papers. Much of the analytic machinery is also available in Chapter 8 of G. Warner's, *Harmonic analysis on semi-simple Lie groups*, vol. II. However Warner's book does not deal systematically with the reductive groups of class $\mathcal{H}$ or include details on the construction of the discrete series characters.

One minor obstacle to the book's usefulness as a reference is two different sets of page numbers. Also, although the list of symbols and index are detailed, in at least one case the page reference was incorrect.

An obvious limitation to Varadarajan's book is that it mentions only the work of Harish-Chandra. One problem that Harish-Chandra's work leaves open is the determination of certain integer constants which appear in formulas for the discrete series characters on regular elements not contained in any compact subgroup. These values are needed when the discrete series characters are used in specific problems of harmonic analysis such as Fourier inversion of orbital integrals. Progress has been made on this problem by T. Hirai, and by W. Schmid and his students, among others **[5]**, **[6]**, **[11b]**.

An important aspect of the discrete series not treated by Harish-Chandra is the problem of concrete realization of the representations. Various geometric realizations, analogous to the Borel-Weil theorem for compact groups, have been established, inspired by a conjecture of Langlands [7], [9], [10], [11a]. The complete problem of existence and realization of the discrete series has recently been treated from a significantly different point of view by M. Atiyah and W. Schmid [1], and the work of Schmid has inspired many recent developments in quite a different spirit than the work of Harish-Chandra.

None of these developments are mentioned in Varadarajan's book. Although a complete treatment of this material would be outside the scope of Varadarajan's book, some remarks and a good list of references would have been useful in bringing it up to date on the theory of the discrete series.

However, recent developments do not diminish the importance of Harish-Chandra's original work. The analytic machinery used such as the Schwartz space and orbital integrals are of independent importance. In addition, Harish-Chandra's construction has been useful in providing an analogy for understanding the representation theory of reductive p-adic groups. Virtually all progress which has recently been made in the study of general reductive p-adic groups has followed Harish-Chandra's program [4b].

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