Suppose $G$ is a finitely generated fuchsian group of the first kind. Let $A(k)$ be the vector space of entire automorphic forms of weight $k$ and

$$A(G) = \bigoplus_{k \geq 0} A(k)$$

the graded ring of automorphic forms. Now $G$ acts on the upper half plane $H_+$ in the usual way. This action has a 'canonical' extension to $H_+ \times \mathbb{C}^*$ via

$$g(z, t) = \left( g(z), \frac{dg}{dz} t \right).$$

**Proposition 1.** $A(G)$ is a graded algebra of finite type. The algebraic set $V = \text{Spec}(A(G))$ is a surface with $\mathbb{C}^*$-action. There is a Zariski open $\mathbb{C}^*$-invariant subset of $V$ which is isomorphic to $(H_+ \times \mathbb{C}^*)/G$.

We thus can use the theory of surfaces with $\mathbb{C}^*$-action to study the structure of the ring of automorphic forms. Now let us suppose that $G$ is a fuchsian group with signature $(g; s; e(1), \ldots, e(f))$ and $V = \text{Spec}(A(G))$. By [7] the singularity of $V$ at $(0)$ has a canonical equivariant resolution. The graph of the resolution is star shaped, of the form

$$\begin{array}{c}
-b \\
-e(1) \\
-e(2) \\
-e(r) \\
\end{array}$$

where $b = 2g - 2 + r + s$.

A first step in understanding the structure of these rings is to find the minimal number of generators, $n$. In [9] we classified all groups with $n \leq 3$. The techniques there are all elementary. The results here are more general since we use...
the more powerful techniques from the theory of singularities of surfaces. The
groups with $n \leq 3$ and $s = 0$ were also classified by Dolgacev [4]. If $g = 0$
and $s > 0$ then the singularity is a rational singularity and one can use the
theory of these singularities to compute $n$. First we let $e = \sum_{i=1}^{r} e(i)$.

**Proposition 2.**  (1) If $s > 1$ then $n = e - r + s - 1$,
(2) If $s = 1$ then $n = e - 3$.

If $g = 0$ and $s = 0$ then the singularity is minimal elliptic [6], hence it
follows directly that:

**Proposition 3.**  (1) if $r > 3$ then $n = \max(3, e - 8)$,
(2) If $r = 3$ and $e(i) > 2$, for all $i$, then

$$n = \max(3, e - 9).$$

(3) If $r = 3$, $e(1) = 2$, $e(2)$, $e(3) > 3$ then

$$n = \max(3, e - 10).$$

(4) If $r = 3$, $e(1) = 2$, $e(2) = 3$, $e(3) > 6$ then

$$n = \max(3, e - 11).$$

To get more information about generators one can use the following de­
scription of $A(k)$ as a vector space of functions on $X$.

**Proposition 4** [8], [9]. Suppose that $p(1), \ldots, p(r) \in X$ are the ellip­
tic points and $q(1), \ldots, q(s) \in X$ are the cusps. Then

$$A(k) = L(kK + k(q((1) + \cdots + q(s)) + \sum_{i=1}^{r} [k(1 - 1/e(i))]p(i)$$

A major tool for stating and proving our results is the Poincaré power
series of the graded algebra $A(G)$. Recall that if $R$ is any graded algebra over a
field $K$ and $M$ is a finitely generated $R$-module, then the Poincaré power series
of $M$ is defined to be

$$P(t) = \sum_{i=0}^{\infty} d(i) t^i$$

where $d(i) = \text{dimension of } R(i) \text{ as a vector space over } K$. Moreover, if $R$ is
finitely generated as an algebra over $K$ then $P(t)$ is a rational function [2]. Now
let $m$ be the maximal ideal of $A(G)$ defined by

$$m = \bigoplus_{k > 0} A(k).$$
Then any basis of $m/m^2$ as a vector space over $\mathbb{C}$ lifts to a minimal set of generators of the algebra $A(G)$. Conversely, every minimal set of algebra generators forms a basis for $m/m^2$. Now $m$ is a graded ideal, hence there is an induced grading on $m/m^2$. Let $p(t)$ be the Poincaré power series of $m/m^2$. Of course this is just a polynomial. The coefficient of $t^i$ in this polynomial is just the number of independent generators of weight $i$.

**Theorem.** If $n > 3$ then

$$p(t) = f(t) + \sum_{i=1}^{r} (t^2 + \cdots + t^{e(0)})$$

where $f(t)$ is given in the table below.

**Signature**

| $s \geq 3$ or $g = 0$ and $s = 2$ | $f(t)$ |
| $s = 2, g \geq 2$ and $1(q(1) + q(2)) = 1$ | $(g + s - 1)t$ |
| $s = 2, g \geq 1$ and $1(q(1) + q(2)) = 2$ | $(g + 1)t + t^2$ |
| $s = 1, g \geq 3, X$ nonhyperelliptic | $gt + 2t^2 + t^3$ |
| $s = 1, g \geq 1, X$ hyperelliptic | $gt + gt^2 + t^3$ |
| $s = 1, g = 0, r \geq 2$ | $-t^2 + (r - 2)t^3$ |
| $s = 0, g \geq 3, X$ nonhyperelliptic | $gt$ |
| $s = 0, g \geq 2, X$ hyperelliptic, $g + r \geq 3$ | $gt + (g - 2)t^2$ |
| $s = 0, g = 1$ and either $e \geq 6$ or $r = 1, e(1) \geq 4$ | $t - t^2$ |
| $s = 0, g = 0, r \geq 4, e \geq 11$ | $-3t^2 + (r - 5)t^3$ |
| $s = 0, g = 0, r = 3, e(i) \geq 3$, for all $i$. | $-3t^2 - 2t^3 - t^4$ |
| $s = 0, g = 0, r = 3, e(1) = 2, e(2), e(3) \geq 4, e \geq 13$ | $-3t^2 - 2t^3 - t^4 - t^5$ |
| $s = 0, g = 0, r = 3, e(1) = 2, e(2) = 3, e(3) \geq 9$ | $-3t^2 - 2t^3 - t^4 - t^5 - t^7$ |

The finite number of signatures which do not appear above all have $A$ generated by 2 or 3 elements. The generators and relations for these rings are listed in [9].

**Corollary.** If we are as above, then $n = e - r + f(1)$.

**Embedding Dimension 4**

The following is a list of all groups whose algebra of automorphic forms is generated by four elements. If the algebra is a complete intersection, that is the ideal of relations is generated by two elements, then we give the degree of the generating relations in the last column.
The groups with \( g = 0 \) are easily found using Propositions 1 and 2. There are 25 signatures that occur in this case.

**BIBLIOGRAPHY**

5. ———, *Automorphic forms and quasi-homogeneous singularities* (preprint).