clarity of the exposition and the precision, which leaves no room for uncertainty. The style has sometimes been characterized as austere or severe. It may, occasionally be also somewhat elliptic. The ideas are presented in a most economical fashion and the author does expect the reader to be able to fill in the more obvious details. This permits him to present the leading ideas in an uncluttered way.

Finally, while the ultimate verdict on the work, like everything human, belongs to history, those of us, who were fortunate enough to have known Harold Davenport, cannot help remembering also the man. While much of what he was–cultured, articulate, logical–is indeed reflected in his work, not everything is. He was generous with his time and enjoyed (or at least seemed to enjoy) showing Cambridge to his guests. While, to judge by his students, his standards must have been very high, he was quite patient with the more common brand of mankind and made genuine efforts to make himself understood by the less sophisticated reader (see, e.g., his book "The Higher Arithmetic"). In fact, this reviewer can recall only one outburst of impatience (or indignation?) of Davenport: it was with mathematicians who claim results, but never publish their proofs, either because they don't have any, or in order to keep their methods as private property of a small group of close collaborators. No names were named.

The reviewer wants to take this opportunity to thank Professor D. J. Lewis for a very helpful letter concerning Davenport which confirmed many and completed some of the reviewer's own recollections.

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Automata-theoretic aspects of formal power series, by Arto Salomaa and Matti Soittola, Texts and Monographs in Computer Science, Springer-Verlag, New York, Heidelberg, Berlin, 1978, x + 178 pp., $16.50.

In the early sixties, stimulated by the discoveries of M. P. Schützenberger, a number of researchers at the University of Paris contributed to a new mathematical approach to the theory of Kleene's regular languages, to Chomsky's context-free languages, and to various other combinatorial questions based on the idea that power-series in noncommuting variables with coefficients taken in a ring or a semiring could play a key role in these fields. A number of simple observations supported this idea, and the relevant one in this review is the fact that the language L generated by a context-free grammar is actually the support of a power-series directly related to the production rules of L. For example, the grammar $\sigma \to \sigma\sigma$, $\sigma \to a\sigma b$, $\sigma \to ab$ produces from the axiom σ a certain set of words on the letters a and b, which is exactly the support of the solution X of the equation $X = X^2 + aXb + ab$ in the semiring of power-series $\mathbf{N}\langle a, b\rangle$. The reader may even verify that the coefficient of $(ab)^3$ in the power-series X is 2; this is also the number of distinct ways of obtaining $(ab)^3$ from σ in the grammar above (ambiguity degree). This is the kind of remark that justifies an approach to regular and context-free languages within the framework of the rigorous formalism of power-series.

After almost twenty years of work a number of deep results have been obtained, mostly by the French School, with a limited impact on computer scientists elsewhere, even though S. Eilenberg's first volume of Automata, Languages and Machines (dealing with regular languages) appeared in 1974.

A. Salomaa and M. Soittola have successfully completed the difficult task of writing a coherent and comprehensive textbook reporting on the major developments in the theory of power-series related not only to regular and context-free languages, but also to stochastic languages and Lindenmayer systems.

There are essentially two types of problems that arise when adopting a power-series approach to the theory of languages. Problems of the first type study the extensions to power-series of the classical results on languages that have originally been obtained through finite state automata, or pushdown acceptors, or grammars. For example, Kleene's theorem says that the class of all the languages accepted by finite automata (recognizable languages) coincides with the smallest class containing the finite languages and closed under the Boolean operations, product, and the star operation (rational languages). A generalized form of this result is established by introducing the appropriate concept of recognizable power-series (matrix representations of power-series), and the concept of rational power-series (solutions of linear equations in power-series semirings). What is gained in the generalizing process might be illustrated briefly by the following example. It is well known that the intersection of a regular language and a context-free language is context-free; the power-series version of this states that the Hadamard product of a rational power-series and an algebraic power-series is an algebraic power-series. This last result generalizes a classical theorem of Jungen on power-series in one variable, and thus provides a surprising link between classical analysis and the theory of languages.

The earliest and perhaps the most important results of this first kind were obtained by Schützenberger between 1959 and 1962 (generalization of Kleene's theorem, matrix representations of power-series, systems of equations, generalization of Jungen's theorem, construction of reduced representations). These results were further developed in a joint-paper with Chomsky that also suggests the idea of rational transduction (under rational transductions the Dyck languages generate all context-free languages). After the works of Elgot and Mezei 1965, Shamir 1967, the first global presentation of the theory of context-free languages from a power-series point of view was written by M. Nivat in 1968, formally introducing power-series transductions and studying their relationship to representations. The work of M. Fliess in 1972, brought further clarifications on the construction of the reduced representation of rational power-series, introduced and developed the use of Hankel matrices for these series. Faithful transductions and cones of power-series were then studied by G. Jacob (1975), and Jacob's work on series with a finite number of distinct coefficients led him recently to show that the finiteness of a finitely generated linear semigroup is decidable.

Problems of the second kind analyze more specifically the passage from languages to power-series and vice-versa. When writing a language as a power-series, the natural domains of the coefficients are either the Boolean semiring $\{0, 1\}$, or the semiring **N**. This leads to questions "à la Fatou"; for example: Is a **Z**-rational power-series with coefficients in **N** an **N**-rational power-series (Soittola proved that the answer is yes for series in one variable). The main contributions in this direction are by Schützenberger 1961, Berstel 1972, Fliess 1972, Soittola 1976. The Skolem-Mahler-Lech theorem on **C**-rational power-series with infinitely many 0 coefficients has served as a source of inspiration for Fatou problems in one variable; in several variables most of the results deal with Fatou extensions of a semiring to another semiring but there are serious difficulties when passing from a semiring to a ring of coefficients. As observed by the authors this approach leads to the arithmetization of language theory, and also provides interesting links with classical analysis.

In addition to the topics discussed above the book contains a power-series approach to stochastic languages (Turakainen 1969, Fliess 1972, Soittola 1976), a study of the concept of density of a regular language (Berstel 1972) and of growth functions of Lindenmayer systems DOL and PDOL (Szilard 1971, Salomaa 1976, Soittola 1976).

The book is divided into three parts: rational series, their applications, and algebraic series. Each chapter contains an ample choice of exercises. The proofs are clearly presented with frequent indications as to how and where the hypotheses are needed. With the exception of a few results quoted without proof, the book is self-contained for the reader who is acquainted with the basics of automata theory and formal languages. The historical and bibliographical remarks have been gathered at the end, which might eventually leave the reader with a feeling of haziness regarding the general progression of ideas. The list of references could be enriched by adding at least [**1**] and [**2**] below ([**2**] shows how polynomials appear in the theory of finite automata), and other applications of power-series in combinatorics, graph theory, coding, bilinear dynamic systems, . . . , could have been at least mentioned by the authors (cf. [**3**]).

Apart from the minor points indicated above, this is a solid textbook that should (1) contribute to establish the fact that the theory of automata and context-free languages leads directly to abstract problems in noncommutative