THE QUEER DIFFERENTIAL EQUATIONS FOR ADIABATIC COMPRESSION OF PLASMA
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This announcement presents some results on the “Queer Differential Equations” (QDE) of adiabatically evolving plasma equilibria. These are nonlinear differential-functional equations of the form \( \Delta \psi = F(V, \psi, \psi', \psi'') \), where \( V = V(\psi) \) is the volume (area) inside the levelsets \( \psi(r) = \psi \), a constant, and the derivatives on the right hand side are with respect to the dependent variable \( V \), e.g., \( \psi'(V) \).

We describe properties of microcanonical averages and their derivatives and a simple example of the nonlinear problem. An existence and uniqueness theorem is given for the associated linearized problem, which is also a functional-differential equation. Finally we will mention an isoperimetric problem related to the geometry of QDE's.

A few of these results are in [1] and will be further expounded in [2]—[4]. For the initial development of this problem and the relevant physics see [5]—[7] and the references therein. For a related problem, see also [8], [9].

Averages. \( S \) is a simple domain in the plane and \( z = \psi(r), r \in \overline{S} \), a surface such that \( \psi \in C^{n, \alpha}(\overline{S}), n \geq 2, 0 \leq \alpha < 1 \). Assume that the level lines \( \psi(r) = \psi \) are simple and \( \nabla \psi = 0 \) at only one point \( r_0 \in S, \psi(r_0) = \psi_0 \). Let \( \partial S \) be the level line \( \psi(r) = \psi_1 (> \psi_0) \). \( V = V(\psi) \) is defined to be the area inside the curve \( \psi(r) = \psi \) and \( \psi(V) \) is the inverse of \( V(\psi) \).

\[
V'(\psi) = \oint_{\psi} |\nabla \psi|^{-1} \, ds
\]

is continuous in the interval \( I = (\psi_0, \psi_1] \).

The function \( V(r) = V(\psi(r)) \) is useful; it is \( C^1 \) in \( \overline{S} \setminus r_0 \).

**Theorem 1.** Let \( \psi \) be as described above and assume that \( r_0 \) is an elliptic critical point. Then \( V(r) = V(\psi(r)) \) is also in \( C^{n, \alpha} \) and \( r_0 \) is an elliptic critical point for \( V = V(r) \).

Consider \( g \in C^{m, \alpha}(\overline{S}) \) and define on \( I \),
Using Green's formula the differentiation with respect to level lines is given by

\[ \tilde{g}(\psi) = \frac{d}{d\psi} g(r) \frac{ds}{|\nabla \psi|^2} = \frac{g}{|\nabla \psi|^2} \frac{ds}{|\nabla \psi|^2} \frac{\partial g(r)}{\partial \psi} \frac{ds}{|\nabla \psi|^2} \]

When \( g = 1 \) this allows us to compute \( V^{(n)}(\psi) \). The integrand in (1) is oscillatory with an unbounded amplitude when \( \psi \to \psi_0 \) (i.e. \( r \to r_0 \)) but the behaviour there can be adequately described by using a coordinate-transformation and Taylor series in the neighborhood of the critical point. In fact we have:

**Theorem 2.** Let \( \psi \in C^{n,\alpha} \) be as in Theorem 1 and \( g \in C^{m,\alpha} \), then \( \tilde{g} \in C^{p,\alpha/2}(I) \), where \( p = \min \{m, n - 1\} \). If \( k \leq p \) and \( r = \max \{ k + 1 - p/2 - \alpha/2, \alpha/2 \} \), then \( (\psi - \psi_0)^{k} \tilde{g}(\psi) \to 0 \) as \( \psi \to \psi_0 \). If \( m = 2k \) and \( n = 2k \), then \( \tilde{g}^{(k)}(\psi) \) can be defined continuously at \( \psi = \psi_0 \).

Finally notice that \( \int \frac{g}{|\nabla \psi|^2} ds = 1 \) and that the microcanonical average of \( g \) is given by

\[ \langle g \rangle_V = \frac{\int g(r) \frac{ds}{|\nabla \psi|^2}}{\int \frac{ds}{|\nabla V|^2}} = \frac{\int g(r) \frac{ds}{|\nabla \psi|^2}}{\int \frac{ds}{|\nabla V|^2}} \]

**Nonlinear problems.** Consider on \( S \)

\[ \Delta \psi = -\psi'' \]

and admit for the moment only solutions with simple level lines. As boundary data we specify \( \psi(r) = \psi_1 \) on \( \partial S \). This is not sufficient for a well-posed problem. Using \( \Delta \psi = \psi' V + \psi'' V|\nabla V|^2 \) and (3) there follows

\[ \Delta V/(1 + |\nabla V|^2) = \frac{df}{d\psi} \frac{V|\nabla V|^2}{\nabla V} = V_1 \]

\[ \psi''/\psi' = -\frac{df}{d\psi} \frac{V(1)}{\psi(1)} = \psi_1 \]

where \( V_1 = V(\psi_1) \) is the area of \( S \). We can now ask if there exists an \( h \) in some properly restricted class of functions, such that the solution of (4), \( V = V(r) \), has simple level lines and satisfies the constraint \( \int_V ds/|\nabla V| = 1 \). The solution of (3) is then given by \( \psi(r) = \psi(V(r)) \) where \( \psi = \psi(V) \) solves (5). It is now clear that other data is needed for (5), and physically it is natural to give \( \psi(0) = \psi_0 \).

No existence theorems are known for classical solutions of (3) for nontrivial geometries, but for simpler QDE's a few results are known.

**Theorem 3.** Let \( \psi \in C^{2,\alpha}(S) \), \( 0 < \alpha < 1 \), have simple level lines and a nondegenerate critical point and solve \( \Delta \psi = F(V, \psi) \), where \( F \in C^{\infty} \). Then \( \psi \in C^{\infty} \).
THEOREM 4. Assume that $T$ is a plane annular domain. Let $\psi \in C^{2,\alpha}(T)$, $0 < \alpha < 1$, have simple level lines and $|\nabla \psi| > 0$ and solve $\Delta \psi = F(V, \psi, \psi')$ where $F \in C^\infty$. Then $\psi \in C^\infty$.

These follow from previous theorems and well-known theorems on the regularity of solutions of elliptic partial differential equations.

THEOREM 5. Let $S$ be convex, let $\Delta \phi = 1$ and $\phi|_{\partial S} = 0$, and let $\phi$ have simple convex level lines with a nondegenerate critical point. Assume that $F = F(V)$ is $C^1$ and $F(0) \neq 0$. Then if $F'$ is sufficiently small,

$$\Delta \psi = F(V), \quad \psi|_{\partial S} = \psi_1$$

has a solution in $C^2$ with the same geometric properties as $\phi$.

This is proven by the iteration $\Delta \psi_m = F(V_{m-1}), V_0(r) = V(\phi(r))$.

Linearized problem. The perturbed problem for $\Delta \psi = F(V, \psi, \psi', \psi'')$ or (3) is also a QDE

$$L_2 \phi = -L_1 \overline{\phi}$$

where $\phi = \partial \psi/\partial t$ and $\overline{\phi}(V) = \langle \phi \rangle_V$. $L_2$ is a second order elliptic operator, $L_2 = \Delta - P$, and $L_1$ is a second order ordinary differential operator. $P$ and the coefficients of $L_1$ are given in terms of $\psi$, which is assumed to have a few derivatives.

What distinguishes the QDE (7) from (3) is that the contours used to determine $\overline{\phi}$ from $\phi$ are now assumed to be given; they are the level lines of $\psi = \psi(r)$.

Then without being too specific,

THEOREM 6. Assume that $L_2$ and $L_1$ are negative definite and have sufficiently smooth coefficients. Let the level lines used to determine $\overline{\phi}$ be simple, have an elliptic critical point and several derivatives. Specify the boundary values $\phi|_{\partial S} = \phi_1$ and $\phi(0) = \phi_0$. Then there exists a unique classical solution for (7).

The proof uses the Green function of $L_2$, $G = G(r, r')$, and its double average $\hat{G}(V, V') = \langle \langle G \rangle_V \rangle_{V'}$, and

THEOREM 7. The double average of a parametrix (or Green function) for a second order elliptic partial differential operator is a parametrix for a second order ordinary differential operator $L_V$, i.e. $L_V \hat{G} = I + A$. $I$ is the identity and $A$ is a compact integral operator.

Using this, some manipulation transforms (7) into a Fredholm integral equation of the second kind for $\overline{\phi}$. 
Isoperimetric problem. The function $K(V) = \langle |\nabla V|^2 \rangle_V$ can be interpreted as the capacity of the (given) distribution of level lines on $S$. It is a purely geometrical quantity.

Let $\phi \in C^1$ and consider the functional

$$\tilde{\Phi}[\psi] = \Phi[V] = \int_0^V \phi(K(V)) dV.$$  

The isoperimetric problem we are interested in is to find the extrema of $\Phi$ subject to the constraint $\oint |\nabla V|^{-1} ds = 1$. In particular, when $\phi(t) = (1 + t)^{-1}$, then the "Euler's equation" for $(8)$ is $(4)$ with an unknown $h$ which has to be chosen so that the constraint is satisfied.

In all the arguments above we have assumed a simple geometry, but non-simple level lines are actually more relevant for physical applications and extensive numerical schemes have been developed by H. Grad and coworkers to study these cases; see the cited references.

The functional in $(8)$ can be reinterpreted so as to include the more complicated geometries. In fact for the special case of $\phi$ mentioned above, finding the supremum of $\Phi[V]$ over a properly constrained class of $V$'s would then correspond to finding the solution of $(4)$ with the simplest geometry.

This variational formulation will be used to study examples of bifurcation, exchange of stability and evolution into more complicated geometries.

REFERENCES