THE DUALITY OPERATION IN THE CHARACTER RING
OF A FINITE CHEVALLEY GROUP
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It is possible (as in [4]) to define a duality operation \( \xi \mapsto \xi^* \) in the ring of virtual characters of an arbitrary finite group with a split \((B, N)\)-pair of characteristic \(p\). Such a group arises as the fixed points under a Frobenius map of a connected reductive algebraic group, defined over a finite field [1]. This paper contains statements of several general properties of the duality map \( \xi \mapsto \xi^* \) and two related operations (see §§2 and 4). The duality map \( \xi \mapsto \xi^* \) generalizes the construction in [2] of the Steinberg character, and interacts well with the organization of the characters from the point of view of cuspidal characters (§6). It is hoped that there is also a useful interaction with the Deligne-Lusztig virtual characters \( R^G_\theta \). Partial results have been obtained in this direction (§5). Detailed proofs will appear elsewhere.

1. Let \( G \) be a finite group with split \((B, N)\)-pair of characteristic \(p\). Let \((W, R)\) be the Coxeter system, and let \( P_j = L_j V_j \) be the standard parabolic subgroup corresponding to \( J \subseteq R \), with \( V_j = O_p(P_j) \) (see [3] for definitions and notations). Let \( \text{char}(G) \) denote the ring of virtual characters of \( G \), and \( \text{Irr}(G) \) the set of irreducible characters of \( G \), all taken in the complex field. For \( J \subseteq R \) and \( \xi \in \text{char}(G) \) define

\[
\hat{\xi}(P_j/V_j) = \Sigma(\xi, \tilde{\lambda})G\lambda
\]

where \( \sim \) denotes extension to \( P_j \) via the projection \( P_j \rightarrow L_j \cong P_j/V_j \), and the sum is over all \( \lambda \in \text{Irr}(L_j) \). Let \( \hat{\xi}(P_j/V_j) \sim \). The duality map is then defined by:

1.2 Definition. \( \xi^* = \Sigma_{J \subseteq R} (-1)^{|J|} \hat{\xi}(P_j)^G \), for all \( \xi \in \text{char}(G) \).

2. The truncation map \( \xi \mapsto \hat{\xi}(P_j/V_j) \) and the map \( \lambda \mapsto \tilde{\lambda}^G \) behave in much the same way as ordinary restriction and induction. The following basic properties follow directly from the structure theorems [3].

2.1 Frobenius reciprocity. Let \( \xi \in \text{char}(G) \) and \( \mu \in \text{char}(L_j) \). Then
\[ (\xi, \tilde{\lambda}^G)_G = (\xi_{(P,J)}, \tilde{\lambda})_{P,J} = (\xi_{P,J/V,J}, \lambda)_{L,J}. \]

2.2 Transitivity. If \( K \subseteq J \subseteq R \), let \( Q_K \) be the standard parabolic subgroup \( P_K \cap L_J \) of \( L_J \) and let \( V_{J,K} = O_p(Q_K) = L_J \cap V_K \). Then if \( \xi \in \text{char}(G) \) and \( \xi \in \text{char}(L_J) \), we have
\[
(\xi_{(P,J/V,J)})(Q_K/V_{J,K}) = \xi_{(P/K/V_K)},
\]
and
\[
(\tilde{\lambda}^{L,J})^G_G = \tilde{\lambda}^G.
\]

2.3 Intertwining number theorem. Let \( \lambda_i \in \text{char}(L_{J_i}) \) for \( i = 1,2 \). Then
\[
(\tilde{\lambda}^G_1, \tilde{\lambda}^G_2)_G = \sum_{w \in W_{J_1,J_2}} (\lambda_1(Q_{K_1}/V_{J_1}), w \lambda_2(Q_{K_2}/V_{J_2,K_2}))_{L,K_1}
\]
where \( W_{J_1,J_2} \) is the set of distinguished \( W_{J_1} - W_{J_2} \) double coset representatives, \( W_{K_1} = W_{J_1} \cap wW_{J_2} \) and \( W_{K_2} = W_{J_2} \cap w^{-1}W_{J_1} \).

2.4 Subgroup Theorem. Let \( \lambda \in \text{char}(L_{J_1}) \). Then
\[
(\tilde{\lambda}^G)(P_{J_2/V_{J_2}}) = \sum_{w \in W_{J_1,J_2}} w^{-1}(\lambda(Q_{K_1}/V_{J_1,K_1}))_{L,J_2}.
\]
Here \( K_1 \) is as in 2.3 (note: \( w^{-1}L_{K_1} = L_{K_2} \)).

3. The results of this section are of independent interest, and are due to Curtis ([4]). They are needed to apply the results of §2 to the duality operation.

3.1. Lemma. Let \( w \in W, wL_{J_2} = L_{J_1}, w\lambda_2 = \lambda_1 \), where \( \lambda_i \in \text{char}(L_{J_i}) \). Then \( \lambda_1^G = \lambda_2^G \).

The idea of the proof is to show that the numbers \( (\tilde{\lambda}^G_1, \tilde{\lambda}^G_2)_G \) are all the same for \( i, j = 1, 2 \). The proof in [3] (for the special case when \( \lambda_1, \lambda_2 \) are cuspidal) can be modified to work in the present situation.

The following is Lemma 2.5 of [4].

3.2. Lemma. Let \( a_{J_2,J_1,K} = |\{w \in W_{J_1,J_2} | W_K = W_{J_1} \cap wW_{J_2}\}|. \)

Then
\[
\sum_{J_2 \subseteq R} (-1)^{|J_2|} a_{J_2,J_1,K} = (-1)^{|K|}.
\]
4. The first main result relates duality and the operations \( \xi \to \xi_{(P_J/V_J)} \) and \( \lambda \to \mathcal{N}^G \). Part (1) is Theorem 1.3 of [4].

**Theorem.**

1. \((\xi_{(P_J/V_J)})^* = (\xi_{(P_J/V_J)})^*_{P_J/V_J}\) for \( J \subseteq R, \xi \in \text{char}(G) \)

2. \((\mathcal{N}^G)^* = (\mathcal{N}^G)^*_{P_J/V_J}\) for \( J \subseteq R, \lambda \in \text{char}(L_J) \).

We provide a sketch of the proof of (2). Let \( J_1 = J \). Using 2.4, 2.2, and then Lemma 3.1 (noting that \( L_K = \omega \mathcal{L}_K \) by Proposition 2.6 of [3]) we have

\[
(\mathcal{N}^G)^* = \sum_{J_2 \subseteq R} (-1)^{|J_2|} \sum_{w \in \mathcal{W}_{J_1,J_2}} \lambda(Q_{K_1/V_J, K_1})^*_{P_J/V_J}
\]

The proof is then completed by applying Lemma 3.2 and 2.2.

**4.2 Theorem.** The map \( \xi \to \xi^* \), from \( \text{char}(G) \to \text{char}(G) \) is an isometry of order two. In particular, \( \xi^{**} = \xi \) and \( \pm \xi^* \in \text{Irr}(G) \), whenever \( \xi \in \text{Irr}(G) \).

In order to prove Theorem 4.2, one first proves that \((\xi_1, \xi_2)^*_G = (\xi_1^*, \xi_2)^*_G\) it then suffices to prove \( \xi^{**} = \xi \). The key is to apply Theorem 4.1 part (1) to the expression for \( \xi^{**} \). We have

\[
\xi^{**} = \sum_{J \subseteq R} (-1)^{|J|} \xi_{(P_J/V_J)}^*_{P_J/V_J}
\]

using 2.2. To finish the proof, note that \( \sum (-1)^{|J|} \) summed over all \( J \) such that \( K \subseteq J \subseteq R \) is zero unless \( K = R \).

5. It is clear that \( \xi^* = (-1)^{|R|} \xi \) for any cuspidal \( \xi \in \text{Irr}(G) \). Thus by applying Theorem 4.1 part (2) we have:

5.1 **Corollary.** Let \( \lambda \in \text{Irr}(L_\lambda) \) be cuspidal. Then \( (\mathcal{N}^G)^* = (-1)^{|J|} \mathcal{N}^G \).

Thus duality permutes (up to sign) the components of \( \mathcal{N}^G \). We can thus determine the "sign" of \( \xi^* \) as follows: \((-1)^{|J|} \xi^* \) is in \( \text{Irr}(G) \) if \( \xi \in \text{Irr}(G) \) is a component of \( \mathcal{N}^G, \lambda \in \text{Irr}(L_J) \) cuspidal. In particular, \( \xi \to \xi^* \) permutes the principal series characters, i.e. the components of \( \mathcal{N}^G, \lambda \in \text{Irr}(L_J) \). A more explicit result is known for the components \( \xi_{\varphi,q} \) of \( 1_{B_1(q)}^G \) in a system of groups \( \{G(q)\} \) of type \( (W, R) \). Specifically, \( \xi^*_{\varphi,q} = \xi_{e\varphi,q} \) where \( e \) is the sign character of \( W ([4]) \).
Finally, consider the case $G = G^F$ where $G$ is a reductive algebraic group and $F : G \rightarrow G$ is a Frobenius map over $F_q$. Let $R^G_T$ denote the Deligne-Lusztig generalized character of $G$ (an $F$-stable maximal torus of $G$, $\theta$ a linear character of $T^F$). It is natural to ask whether

\[(R^G_T \theta)^* = \pm R^G_T \theta\]

holds. The following suggests the answer is yes.

\[(R^G_T \theta)^*(s) = \pm R^G_T \theta(s)\]

for semisimple elements $s$ of $G$. The ± sign in 5.3 does not depend on the particular element $s$ of $G$. The proof of 5.3 uses several results of [5]. (Note added in proof: The conjecture 5.2 has been proved by G. Lusztig.)

5.4 Example. Let $G = G^F$ as above, with (relative) Coxeter system $(W, R)$. Let $V$ be the set of unipotent elements of $G$ and let $\epsilon_V$ be the characteristic function of $V$. A recent result of Springer (Theorem 1 of [6])\(^1\) shows

\[\epsilon_V = q^d \sum_{J \subseteq R} (-1)^{|J|} |P_J|^{-1} 1^G_J\]

where $d = \dim(G/B)$, $B$ a Borel subgroup of $G$. Applying Theorems 4.1 and 4.2 we have:

5.5 Theorem. (1) $\epsilon_V^* = (q^d/|G|) \rho_G$ where $\rho_G$ is the regular character of $G$.

(2) For $\xi \in \text{Irr}(G),$

\[\frac{1}{\xi(1)} \sum_{v \in V} \xi(v) = q^d (\xi^*(1)/\xi(1)).\]

(3) For $\xi \in \text{Irr}(G), |\xi^*(1)|_{p'} = \xi(1)_{p'}$, where $p$ is the characteristic of $F_q$ and $n_{p'}$ is the $p'$ part of $n$.

(4) For $\xi \in \text{Irr}(G), 1/\xi(1) \sum_{v \in V} \xi(v)$ is, up to sign, a power of $p$.

Part (4) of Theorem 5.5 confirms a special case of a conjecture of Macdonald (see [6]), namely the case when $q = p$ is prime.

REFERENCES


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