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The theory of stochastic processes, by I. I. Gihman and A. V. Skorohod, translated from the Russian by S. Kotz; Volume II, Grundlehren der Mathematischen Wissenschaft, vol. 218, Springer-Verlag, New York-Heidelberg-Berlin, 1975, vii + 441 pp., $48.20; Volume III, Grundlehren der Mathematischen Wissenschaft, vol. 232, Springer-Verlag, New York-Heidelberg-Berlin, 1979, viii + 387 pp., $47.80.

Modern Markov process theorists can be divided, more or less accurately, into two camps: descendants of the Hunt-Dynkin-Blumenthal-Getoor school and descendants of the Lévy-Itô school. Although the analogy is not entirely precise, the distinction between these two camps is comparable to the distinction between students of locally compact groups and specialists in Lie groups. Members of the H.D.B.G. camp arrive on the scene already equipped with a transition probability function satisfying certain apparently benign hypotheses. (Where their transition function comes from or how they verified that it satisfies their hypotheses is often somewhat obscure.) On the basis of their hypotheses, they proceed to show that the Markov process determined by their transition function possesses a surprising number of properties in common with Brownian motion. The technique that they use involves an intricate mixture of functional analysis, potential theory, and measure theory; and by this time the state of their art has achieved a high level of sophistication. The problem with their theory is that it remains difficult to determine in specific examples whether the theory applies and, even when it does apply, whether it answers the questions that need answering.

The basic reason why the general theory of the H.D.B.G. camp does not lend itself easily to application is that, in practice, one is seldom presented with an explicit transition probability function. Instead, one is usually presented with an implicit description of the transition mechanism. Typically, all that is known is approximately what the process will do between times t and $t + h$ (h being very small) given what has happened up until time t. If one wants to use the H.D.B.G. theory, one must first construct from such an implicit description a transition probability function. This is sometimes possible. For example, in the case of diffusion processes, it is a relatively easy task to incorporate the infinitesimal description of the transition mechanism into a parabolic differential equation: either the "backward" or "forward" (Fokker-Planck) equation. Once this has been done, one can call upon all the heavy artillery from P.D.E.'s to construct the desired transition function. Of course, such a procedure does not shed very much light on the structure of the diffusion under consideration; it simply brings one to the place where the machinery of the H.D.B.G. school can be mobilized.

The point of departure for disciples of the L.I. schools is the observation that a great many Markov processes resemble, at least in the small, certain "standard processes". (These standard processes are usually the independent increment processes.) For instance, a diffusion $x(t)$ is a process such that, for small $h > 0$, $x(t + h) - x(t)$ given $x(s)$, $0 \leq s \leq t$, is (*apart from* $o(h)$-*terms*) a Gaussian variable with mean and variance equal to h times functions of $x(t)$. Itô's stochastic differential equations are the precise formulation of this idea. More generally, $x(t + h) - x(t)$ given $x(s)$, $0 \leq s \leq t$, can often be approximated by the increment of a standard process depending on $x(t)$. The determination of what standard processes make up the process under consideration can be read off directly from the description of the infinitesimal transition mechanism. Thus the L.I. approach has the advantage that it takes off from a point which is more likely to be accessible directly from the given description of the process. Furthermore, the L.I. approach, when it works, eliminates the necessity of first constructing the transition probability function and only then getting at the process. Finally, for the probabilist, the L.I. theory is the intuitively appealing one since a probabilist presumably feels quite comfortable with the standard processes and is therefore happy with the description of his general process provided by the L.I. school.

Unfortunately, not every Markov process is amenable to the L.I. techniques. For example, the processes associated with nondegenerate divergence form operators having nondifferentiable coefficients defy treatment by the L. I. theory; although, using the beautiful results from the analytic theory of such operators, one can fit them into the H.D.B.G. scheme. Thus, there undeniably is an important place for both schools.

Volumes II and III of *The theory of stochastic processes* by I. I. Gihman and A. V. Skorohod comprise an up-to-date account of the theory of Markov processes. Volume II is written from the H.D.B.G. perspective; whereas volume III follows the L.I. line and is obviously greatly influenced by the ideas of P. A. Meyer and H. Kunita and S. Watanabe. Both volumes are written in the modern style; that is, they progress from the most general to the specific. In the case of volume II, this means that the first two chapters (about 40% of the book) very much resemble the magnificent but formidable book by E. B. Dynkin: *Markov processes*, Springer-Verlag (1965). In the case of volume III, it means that they begin with the general structure theory for quasi-martingales. In spite of their being written in the modern style, both volumes give a quite leisurely presentation of the subjects and contain many honest, and often successful, attempts to provide the reader with some insight into the material. Thus, for example, already in volume II the reader is introduced to the concept of infinitesimal characteristics and is shown how these are reflected by the processes under study via Kolmogorov's "backward" and "forward" equations. A second example is the extremely careful treatment in volume II of independent increment processes in such a way that the structure theory for quasi-martingales in volume III comes as no great surprise to the reader. The result of their explanatory efforts is that Gihman and Skorohod have produced a book which is far more accessible to the novice than are most books written at so advanced a level.

Both volumes contain a great deal of material which is often slighted in other treatments. For instance, volume II contains an excellent and detailed discussion of Dynkin's characteristic operator and its relationship to the infinitesimal generator. The same volume presents the theory of semi-Markov processes and introduces the reader to branching Markov processes (although this latter topic is given rather short shrift). In volume III, stochastic differential equations are done as much as possible in the context of Itô processes, thus handling equations with delay simultaneously with the Markov case. Also, considerable stress is put on the distinction between weak and strong existence and uniqueness of solutions to stochastic differential equations; in particular, the contributions to this topic due to Girsanov and Skorohod are explained.

Having also reviewed volume I of Gihman and Skorohod's treatise (cf. Bull. Amer. Math. Soc. **82** (1976)), I feel compelled to summarize my impressions of this *magnum opus* now that I have seen volumes II and III. I suggested in my review of volume I that if these authors' intention was to write the "Dunford and Schwartz" of stochastic process theory, then they had fallen short of their goal. After reading volumes II and III, I now realize that they had no such idea in mind; the subject of stochastic processes remains in far too incomplete a state to admit a definitive text. Like Doob in his renowned book on the same topic, Gihman and Skorohod have presented in their book a broad view of an imperfect subject. By the time that the subject has been perfected, chances are that no one will any longer want to read a book about it. In the meantime, Gihman and Skorohod have done an excellent job of presenting the theory in its present state of rich imperfection.

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