Continuous maps from the interval [0, 1] to itself have been studied for some time as simple models of dynamical systems with discrete time. In particular, the map \( x \mapsto 1 - 2|x - \frac{1}{2}| \) has no stable periodic orbit on [0, 1]. In the paper \([1]\) we show that such behaviour is very common among the members of a parametrized family of maps which contain a quadratic critical point.

Let \( 0 < \delta < \frac{1}{2} \) and define the map \( f_\delta : [0, 1] \to [0, 1] \) by

\[
f_\delta(x) = \begin{cases} 
1 - \delta - (x - \frac{1}{2})^2/\delta & \text{for } x \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta], \\
1 - 2|x - \frac{1}{2}| & \text{for } x \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1].
\end{cases}
\]

**Figure 1.** The function \( f_\delta \) for \( \delta = 0.15 \).
In [1] we construct a Lebesgue measurable subset $M$ of $[0, \frac{1}{2})$ with positive measure having the properties:

(a) $M$ gets thicker near $\delta = 0$, i.e., Leb. meas. $(M \cap [0, \delta])/\delta > 1 - 1/\log \delta$, for small $\delta > 0$.

(b) If $\delta \in M$, the map $f_{\delta}$ has no stable periodic orbit.

(c) If $\delta \in M$, $f_{\delta}$ is topologically conjugate to a piecewise linear map $g_{\tau(\delta)}$: $x \rightarrow \tau(\delta) \cdot (1/2 - |x - \frac{1}{2}|)$, with $\tau > 2^{\frac{1}{6}}$.

REMARKS. (1) (a), (b), and (c) show the abundance of aperiodic behaviour among the $f_{\delta}$, when $\delta$ is near 0. This settles an old question about maps on the interval.

(2) (c) implies sensitive dependence on initial conditions in the sense of Guckenheimer [2].

(3) The often-studied maps $x \rightarrow 4sx(1 - x)$ with $s = 1 - \delta^2/8$ are conjugated (through $x = \sin^2(\pi y/2)$) to $y \rightarrow 1 - \delta - (2/\delta)(y - \frac{1}{2})^2 + O(\delta^2)$ (for $y$ near $\frac{1}{2}$). This motivates our choice of $f_{\delta}$.

**SKETCH OF THE PROOF FOR THEOREM 1.** The proof relies on a representation of a number in the interval $E_{\delta} = [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ by a sequence $(n_t, A_t, e_t)_{t \in \mathbb{N}}$, where $n_t \in \mathbb{N}, A_t \in \mathbb{N}$ and $e_t = \pm 1$. The representation is defined recursively. Let $x_0 \in E_{\delta}$ be the number to be represented, and assume $(n_t, A_t, e_t)_{t=1,\ldots,j}$ have been computed together with $x_j \in E_{\delta}$. $n_{j+1}$ is defined as the smallest integer such that $f_{\delta}^{n_{j+1}}(x_j)$ is in $E_{\delta}$. $x_{j+1}$ is equal to $f_{\delta}^{n_{j+1}}(x_j)$, $A_{j+1}$ is the integer part of

$$2^{n_{j+1}-1}(\delta + (x_j - \frac{1}{2})^2/\delta) \text{ and } 2^{n_{j+1}-1}(\delta + (x_j - \frac{1}{2})^2/\delta) - A_{j+1}$$

is equal to $e_{j+1}x_{j+1}$. Some parts of the proof of Theorem 1 are reminiscent of the small divisor problem. In particular, if for some $\delta$, $x_j = \frac{1}{2}$, then $\frac{1}{2}$ is a stable periodic point. Such a $\delta$, together with a small $\delta$ interval around it, does not belong to $M$ and can be considered as a resonance. The measure of $M$ is investigated using a lower bound on $dx/d\delta$ when $\delta$ is in $M$.

**REFERENCES**


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