A POINCARÉ-HOPF TYPE THEOREM
FOR THE DE RHAM IN Variant
BY DANIEL CHESS

The Poincaré-Hopf theorem relates the Euler-characteristic of a manifold to the local behavior of a generic vector-field on the manifold in a neighborhood of its zeroes. As a corollary of this, by taking the gradient, one can calculate the Euler-characteristic of a manifold from a local knowledge of a generic map to $R^1$ around its singular points. We prove an analogue of this theorem for calculation of the de Rham invariant of $4k + 1$ dimensional orientable manifolds from a map to $R^2$.

For $4k + 1$ dimensional orientable manifolds we have the de Rham invariant $d(m)$. This invariant is

(a) the rank of the 2-torsion in $H_{2k}(M)$,
(b) $\hat{\chi}_0(M) - \hat{\chi}_2(M)$ mod 2 where $\hat{\chi}_F(M)$ is the semicharacteristic of $M$ with coefficients in $F$,
(c) $d(M) = [w_2w_{4k-1}(M), [M]] = [v_{2k}s^1v_{2k}(M), [M]]$, where $w_i(M)$ is the $i$th Stiefel-Whitney class and $\nu_i$ is the $i$th Wu class of $M$.

For the equivalence of these definitions see [L-M-P]. The de Rham invariant is important in the theory of surgery; see [M] or [M-S].

Definition of the local invariant. Let $M^m, N^n$ be $C^\infty$ manifolds. Let $C^\infty(M, N)$ be the space of $C^\infty$ maps from $M$ to $N$ topologized with the $C^\infty$ topology. Within $C^\infty(M, N)$ we have a dense (in fact residual) subset $G(M, N)$ of maps which are generic in the sense of Thom-Boardman [B] and satisfy the normal crossing condition [G-G]. This second condition is essentially that $f$ is in general position as a map of its singularity submanifolds to $N$.

Let $f \in G(M, R^2)$; then $df$ is of rank 2 except on a collection of disjoint closed curves in $M$, the singular set of $f, S_1(f)$. At points of $S_1(f), df$ is of rank 1. Restricted to $S_1(F) f$ is an immersion except at a finite set of points, $S_{1,1}(f)$, the cusp points of $f$. $S_1(f) - S_{1,1}(f) = S_{1,0}(f)$ is the set of fold points of $f$. Suppose $x \in S_{1,0}(f)$ then we can choose coordinates $x_1, \ldots, x_n$ around $x$ and coordinates $y_1, y_2$ around $f(x)$ so that

$$f(x_1, \ldots, x_n) = (x_1, x_2^2 + x_3^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2).$$
Similarly if \( x \) is a cusp point we can choose coordinates so that

\[
f(x_1, \ldots, x_n) = (x_1, x_2^3 + x_1 x_2 + x_3^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2).
\]

Now we quote a result from [L].

**Theorem.** Let \( M^m, m > 2 \) be of even Euler characteristic; then given \( f \in G(M, R^2) \), \( f \) is homotopic to an \( f_1 \) in \( G(M, R^2) \) with no cusp points.

In the case that \( f \in G(M, R^2) \) has no cusps \( f|S_1(f) \) is an immersion. The normal crossing condition guarantees that \( f(S_1(f)) \) crosses itself in a finite number of double points with no triple points. Let

\[
V(f) = \{ y \in R^2 | f^{-1}(y) \cap S_1(f) = 2 \text{ points} \}.
\]

Let \( N(S_1(f), M) \) be the normal bundle to \( S_1(f) \) in \( M \) and let \( G \) be the bundle over \( S_1(f) \) defined by the following exact sequence:

\[
T(M)|S_1(f) \xrightarrow{df} f^*T(R^2) \rightarrow G \rightarrow 0
\]

where if \( M \) is a manifold \( T(M) \) denotes its tangent bundle. Use of the (second) intrinsic derivative [L], [B] allows definition of a symmetric bilinear form

\[
B: N(S_1(f), M) \otimes N(S_1(f), M) \rightarrow G
\]
on \( N(S_1(f), M) \) with values in \( G \). \( B \) is nondegenerate on \( S_{1,0}(f) \) and has one-dimensional kernel on \( S_{1,1,0} \). Given \( x \in S_{1,0}(f) \) and a choice of an orientation of \( G_x, B_x \) is a nondegenerate bilinear form on \( N(S_1(f), M)_x \). Define the absolute index \( a(x) \) at \( x \) by \( a(x) = \min(\text{index}(B_x), m - 1 - \text{index}(B_x)) \). Note that \( a(x) \) is independent of the choice of orientation of \( G_x \).

Explicitly suppose \( C_j \) is a component of \( S_1(f) \) with no cusps, then \( f|C_j \) is an immersion so \( N(f(C_j), R^2) \) the normal bundle to \( F(C_j) \) in \( R^2 \) is well defined, isomorphic to \( G \), and trivializable. Let \( T: N(f(C_j), R^2) \rightarrow R^1 \) be a trivialization. Let \( D(C_j, M) \) be a choice of normal disc bundle to \( C_j \) in \( M \). By a slight abuse of notation we consider \( f: D(C_j, M) \rightarrow N(f(C_j), R^2) \); let \( g_x: D_x(C_j, M) \rightarrow R^1 \) denote the function

\[
g_x = T \circ f: D_x(C_j, M) \rightarrow R^1.
\]

Then \( \{g_x | x \in C_j \} \) is a differentiable family of functions on the fibers of \( D(C_j, M) \), each \( g_x \) has a Morse singularity at \( x \) and the form \( B_x \) is given by \( d^2 g_x \).

Let \( M^m \) be orientable with \( n = 2k + 1 \); then \( M \) has zero Euler-characteristic so use the theorem of Levine to choose \( f \in G(M, R^2) \) with no cusps. Let \( C_j \) be a component of \( S_1(f) \); as \( M \) is orientable \( N(C_j, M) \) is trivializable. Furthermore the choice of trivialization \( T \) above makes \( B \) a nondegenerate symmetric bilinear form on \( N(C_j, M) \). Thus the structure group of \( N(C_j, M) \) is given a reduction to \( O^+(p, m - p - 1) \), the orientation preserving components of
A POINCARÉ-HOPF THEOREM

1033

$O(p, m - p - 1)$. For $p \neq 0, m - 1, O^+(p, m - p - 1)$ has two components. Define $i(C_j) \in \mathbb{Z}/2\mathbb{Z}$, the index of $C_j$, by $i(C_j) = 0$ if and only if $N(C_j, M)$ is the trivial $O^+(p, m - p - 1)$ bundle and $i(C_j) = 1$ if and only if $N(C_j, M)$ is the nontrivial $O^+(p, m - p - 1)$ bundle. Note that $i(C_j)$ is independent of all choices of trivialization and orientations.

Define $\tau(f) \in \mathbb{Z}/2\mathbb{Z}$ by

$$\tau(f) = \sum i(C_j), \quad C_j \text{ a component of } S_1(f).$$

Define $r(f) = |V(f)| \mod 2$.

Statement of results.

**Proposition A.** $r(f) = t(f) + \tau(f)$ is independent of the choice of $f \in G(M, R^2)$ without cusps, so one can define $r(M) = r(f), f \in G(M, R^2)$ without cusps.

**Comment.** One can, with a little more effort, still define $r(f)$ for $f \in G(M, R^2)$ even when $f \in G(M, R^2)$ has cusps. However in this case $r(f)$ is no longer independent of $f$.

**Proposition B.** $r$ is a homomorphism from oriented cobordism to $\mathbb{Z}/2\mathbb{Z}$; that is

(a) If $[M] = [N]$ in $\Omega_{2k+1}$ then $r(M) = r(N)$,

(b) $r(M_1 \cup M_2) = r(M_1) + r(M_2)$.

Let $\chi(M)$ be the Euler characteristic of $M$ reduced mod 2.

(c) $r(M^{2k+1} \times N^{2p}) = r(M^{2k+1}) \cdot \chi(N^{2p})$.

**Proposition C.**

(a) Let $M^{4k+1}$ be an orientable manifold then $r(M) = d(M)$.

(b) Let $M^{4k+3}$ be an orientable manifold then $r(M) = 0$.

Thus Proposition C gives a way of determining the de Rham invariant, which is intersection theoretic in character, from the local behavior of a map $M$ to $R^2$ around its singular set. It is illuminating to consider such an $f$ as a pair of Morse functions in general position with respect to each other.

**Sketch of proofs.** Proposition A is proved by a careful analysis of a homotopy $F$ from $f_0$ to $f_1$ where $f_0$ and $f_1$ are different choices of $f$ on $M^{2k+1}$. We can take $F \in G(M \times I, R^2 \times I)$. First we reduce to the case that $F$ has no dovetail singularities. In this case $S_1(F)$ is an embedded surface in $M \times I$ intersecting $M \times \{0\}$ and $M \times \{1\}$ normally in $S_1(f_0)$ and $S_1(f_1)$. On the interior of this surface we have circles of cusp points separating the surface into regions of constant absolute index. Let $R_p$ be the union of the regions of absolute index $p$. Let $i(R_p) = \sum i(C)$ a component of $\partial(R_p)$, then analysis of the cusp...
singularity yields the equation

\[ \sum_{i=0}^{k} i(R_p) = \tau(f_0) + \tau(f_1). \]

For \( p \neq k \) it is straightforward to prove \( i(R_p) = 0 \). As in the case that \( F \) has no dovetails we have

\[ t(f_0) = t(f_1) \mod 2. \]

Proposition A reduces to showing \( i(R_k) = 0 \). This is easy for \( k \) odd, but subtler for \( k \) even. For \( k \) even we prove

**Lemma.** Let \( P \) be a component of \( R_k \); then \( P \) is a closed surface \( P^1 \) minus a collection of discs and \( P^1 \) is of even Euler characteristic. From this fact it follows from \( i(R_k) = 0 \).

Proposition B is proved by first observing that \( r(M) \) remains invariant if \( M \) is cut open and repasted along a codimension 1 submanifold of the form \( S^1 \times F \) by a pasting \( \phi: S^1 \times F \rightarrow S^1 \times F \) with \( \phi(x \times F) = x \times F \). Given this observation the results of [A] allow immediate demonstration of bordism invariance. For the relation of cutting and pasting and cobordism see [K-K-N-O].

Proposition C follows from explicit construction of the examples (using, for instance, (c) of Proposition B) in each dimension \( 4k + 1 \) on which \( r \) and \( d \) agree and are nonzero. This, in addition to the result of [Br] that \( d(M) \) vanishes if and only if \( [M] \) has a representative fibered over the two-sphere, is enough to show \( r(M^{4k+3}) = 0 \) use the results of [A-K] to choose a representative of \( [M] \) fibered over \( S^2 \). On such a representative \( r(M) \) is zero by the observation of the previous paragraph.

**References**


[A-K] J. C. Alexander and S. M. Kahn, *Characteristic number obstructions to fibering oriented and complex manifolds over surfaces*, Univ. of Maryland, College Park, Maryland (Preprint).


A POINCARÉ-HOPF TYPE THEOREM


DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

Current address: Courant Institute of Mathematical Sciences, New York University, New York, New York 10012