BLOCKS WITH CYCLIC DEFECT GROUPS IN $GL(n, q)$

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Let $G$ be a finite group and $B$ an $r$-block of $G$ with cyclic defect group $R$. The decomposition of the ordinary characters in $B$ into modular characters is described by the Brauer tree $T$ of $B$. The problem of determining the Brauer trees for finite groups of Chevalley type was proposed by Feit at the 1979 AMS Summer Institute. Our result is a necessary step in this problem: If $G = GL(n, q)$ and $r$ is an odd prime not dividing $q$, then $T$ is an open polygon with its exceptional vertex at one end. The proof also shows an interesting fit of the modular theory for such primes $r$ with the underlying algebraic group, the Deligne-Lusztig theory, and Young diagrams.

Because $R$ is a cyclic defect group, $R$ has the form

$$R = \begin{pmatrix} I_l & 0 \\ 0 & R_1 \end{pmatrix},$$

(1)

where the elementary divisors of a generator of $R_1$ are, say, $m$ copies of an irreducible polynomial of degree $d$ over $F_q$. By (1) the structure of $C = C_G(R)$ is

$$C = \begin{pmatrix} C_0 & 0 \\ 0 & C_1 \end{pmatrix},$$

(2)

where $C_0 \cong GL(l, q)$ and $C_1 \cong GL(m, q^d)$. The normalizer $N = N_G(R)$ is then obtained by adjoining to $C$ an element $t$ of the form

$$t = \begin{pmatrix} I_l & 0 \\ 0 & t_1 \end{pmatrix},$$

where $t_1$ induces a field automorphism of order $d$ on $C_1$.

By Brauer's First Main Theorem $B$ corresponds to a block $B_C$ of $C$ with defect group $R$, where $B_C$ is determined up to conjugacy in $N$. Let $E$ be the stabilizer of $B_C$ in $N$, so $e = [E: C]$ is then the inertial index of $B$. Let $\Lambda$ be a set of representatives for the orbits of $E$ on the set of nontrivial irreducible characters of $R$. In the Brauer-Dade theory [1] the exceptional characters $\chi_\lambda$ in $B$ are labeled by $\lambda$ in $\Lambda$, the nonexceptional characters $\chi_i$ in $B$ by $i = 1, 2, \ldots, e$, and the $e + 1$ vertices of $T$ by $\chi_1, \chi_2, \ldots, \chi_e$, exc.

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The block $B_C$ decomposes by (2) as $B_C = B_{C_0} \times B_{C_1}$, where $B_{C_0}$ is a block of $C_0$ of defect 0 and $B_{C_1}$ is a block of $C_1$ with defect group $R_1$. Since $R_1 \leq Z(C_1)$, a theorem of Reynolds [6] implies that the characters $\theta_\lambda$ in $B_{C_1}$ can be labeled by the irreducible characters $\lambda$ of $R$ so that $\theta_1$ is the unique $r$-rational character in $B_{C_1}$ and so that $\theta_\lambda(xy) = \lambda(x)\theta_1(y)$ for any $x$ in $R_1$ and any $r'$-element $y$ of $C_1$.

Let $R_C^G(\theta)$ be the virtual character of $G$ associated to the irreducible character $\theta$ of $C$ by the Deligne-Lusztig theory. Thus $R_C^G(\theta)$ is an element of the Grothendieck ring of representations of $G$ over $\overline{Q}_l$. (We have modified the notation of [2], [4]. We should strictly write $R_C^G(\theta)$, where $G$ is the algebraic group $GL(n, \overline{F}_q)$, $F$ is a Frobenius endomorphism of $G$ with $G = \overline{G}^F$, $\overline{C}$ is a regular subgroup of $\overline{G}$, and $C = \overline{C}^F$.)

**Proposition 1.** There is a labeling of the $X_\lambda, \lambda \in \Lambda$, and signs $e_0, e_1, \ldots, e_e$ such that

$$R_C^G(\theta_0 \theta_\lambda) = e_0 X_\lambda \quad \text{for } \lambda \in \Lambda,$$

$$R_C^G(\theta_0 \theta_1) = e_1 X_1 + e_2 X_2 + \cdots + e_e X_e.$$

The signs $e_i$ and the generalized decomposition numbers of $B$ corresponding to a generator $x$ of $R$ are related by

$$e_i = d^x(X_i, \theta_0 \theta_1) \quad \text{for } 1 \leq i \leq e,$$

$$e_0 \sum_{g \in E/C} \lambda^g = d^x(X_\lambda, \theta_0 \theta_1) \quad \text{for } \lambda \in \Lambda.$$

A classification of the irreducible characters of $G$ has been given by Green [3], and can be restated in the language of [5] as follows: Each irreducible character $\theta$ of $G$ corresponds to an ordered pair $(s, \xi)$, where $s$ is a semisimple element of $G$ and $\xi$ is a unipotent irreducible character of $C_G(s)$. The correspondence is given by $\theta = \pm R_C^G(s)(\xi\eta)$, where $\eta$ is the linear character of $C_G(s)$ dual to $s$. (The dual $\eta$ of $s$ is defined by fixing an isomorphism of $F^*$ into $\mathbb{Q}$. Then $\text{Hom}(\overline{L}^F/\overline{L}, \overline{L})^F \approx (Z(\overline{L}))^F$ for any regular subgroup $\overline{L}$ of $\overline{G}$. In particular, if $s$ is a semisimple element of $G$ and $\overline{L} = C_G(s)$, then $s$ is in $(Z(\overline{L}))^F$ and thus determines a linear character $\eta$ of $C_G(s)$.)

Let $(s_i, \xi_i)$ be a pair corresponding to the irreducible character $\theta_i$ of $C_i$ for $i = 0, 1$. Thus

$$\theta_i = \pm R_{L_i}^C(\xi_i \eta_i),$$

where $L_i = C_{C_i}(s_i)$ and $\eta_i$ is the linear character of $L_i$ dual to $s_i$. In the case $i = 0$, $\xi_0$ has defect 0; in the case $i = 1, L_1$ is a torus of order $q^{dm} - 1$ in $C_1$ and $\xi_1$ is the 1-character of $L_1$. Let $s = s_0 s_1$ and $K = C_G(s)$. It then follows that $C_K(R) = C_C(s) = L_0 \times L_1, N_K(R) = (C_K(R), t^f)$, and $|N_K(R)/C_K(R)| = e$, where $d = ef$. In particular, if $B_{L_0}$ is the block of $L_0$ containing $\xi_0$ and $B_{L_1}$ is
the principal block of $L_1$, then $(B_{L_0} \times B_{L_1})^K$ is a block $B_K$ of $K$ with defect group $R$ and inertial index $e$. Let $\psi_\lambda, \lambda \in \Lambda$, be the exceptional characters in $B_K$, and $\psi_1, \psi_2, \ldots, \psi_e$ the nonexceptional characters in $B_K$. The $\psi_1, \psi_2, \ldots, \psi_e$ are then unipotent. Let $\eta$ be the linear character of $K$ dual to $s$.

**Proposition 2.** There is a labeling of the $\psi_1, \psi_2, \ldots, \psi_e$ and signs $\delta_0, \delta_1, \ldots, \delta_e$ such that

\[
R^K_G(\delta_0 \psi_\lambda \eta) = e_0 \chi_\lambda, \quad \lambda \in \Lambda, \\
R^K_G(\delta_i \psi_i \eta) = e_i \chi_i, \quad 1 \leq i \leq e.
\]

In particular, $R^K_G$ induces a 1-1 correspondence between the nonexceptional characters of $B$ and the unipotent characters in $B_K$.

The group $K$ is a direct product $K = K_1 \times \cdots \times K_t$, where $K_i$, say, is isomorphic to $GL(n_i, q^{d_i})$. Let $B_K = B_{K_1} \times \cdots \times B_{K_t}$ be the corresponding decomposition of $B_K$. We may label these so that $B_{K_i}$ has defect 0 for $i > 1$. Then $B_{K_1}$ has defect group $R$. The unipotent characters of $K_1$ are in natural 1-1 correspondence with the irreducible characters of the Weyl group $W_1$ of $K_1$, and since $W_1$ is the symmetric group on $n_1$ symbols, the unipotent characters of $K_1$ are thus naturally labeled by partitions of $n_1$. Since the characters in $B_K$ are products of a fixed character of $K_2 \times \cdots \times K_t$ by the characters in $B_{K_1}$, we may also label the nonexceptional characters in $B_K$ by the partitions $\mu_1, \mu_2, \ldots, \mu_e$ of $n_1$ labeling the unipotent characters in $B_{K_1}$. Thus we write $\psi_\mu_i$ for $\psi_i$.

**Proposition 3.** Each partition $\mu_i$ has a unique $e$-hook $\nu_i$ and the hooks $\nu_1, \nu_2, \ldots, \nu_e$ are pairwise distinct. The $e$-core $\mu_0$ of $\mu_i$, obtained by deleting $\nu_i$ from $\mu_i$, is the same for all $i$; moreover, $\mu_0$ has no $e$-hooks. The $\mu_i$ account for all partitions $\mu$ of $n_1$ with a unique $e$-hook and $e$-core $\mu_0$.

Let the partitions $\mu_i$ be arranged so that the $e$-hook $\nu_i$ of $\mu_i$ has leg length $i - 1$. By our labeling the nonexceptional characters in $B$ are of the form $\chi_i = R^K_G(\delta_i e_i \psi_\mu_i \eta)$.

**Proposition 4.** If the $\chi_i$ are labeled as above, then the signs $\delta_i$ are given by $\delta_i = (-1)^{i-1}$. Moreover, $R^K_G$ induces a graph isomorphism of the Brauer trees of $B_K$ and $B$. The tree of $B$ is the open polygon.

\[
\begin{array}{cccccc}
\chi_1 & \chi_2 & \chi_3 & \cdots & \chi_e & \text{exc}
\end{array}
\]

**References**


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