ON THE LOCAL MONODROMY OF A VARIATION OF HODGE STRUCTURE

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Associated to a variation of polarized Hodge structure there is a period mapping \( \psi: S \to \Gamma \setminus \mathcal{D} \), where \( S \) is the parameter space and \( \Gamma \setminus \mathcal{D} \) denotes the corresponding modular variety of polarized Hodge structures (the primary example to keep in mind is that arising from a family of smooth projective varieties parametrized by \( S \)) [3], [4]. The local study of the singularities of \( \psi \) ([5]) reduces to the case when \( S = (\mathbb{A}^*)^l \times \mathbb{A}^m \), a product of punctured disks and disks.

Given a lifting \( \widetilde{\psi}: U^l \times \Delta^m \to D \) (\( U = \) upper half-plane) of \( \psi \) to the universal covering of \( S \) there are monodromy transformations \( \gamma_1, \ldots, \gamma_l \in \Gamma \) such that

\[
\widetilde{\psi}(z_1, \ldots, z_i + 1, \ldots, z_l; w_1, \ldots, w_m) = \gamma_i \widetilde{\psi}(z_1, \ldots, z_i, \ldots, z_l; w_1, \ldots, w_m).
\]

These \( \gamma_i \)'s, which are quasi-unipotent automorphisms of the \( \mathbb{C} \)-vector space \( H \) underlying the variation, provide important invariants of the singularities of \( \psi \). In particular, in the single variable case \( (l = 1, m = 0) \) a central role is played by the monodromy weight filtration \( W_\bullet = W_\bullet(N) \) of the nilpotent transformation \( N = \log \gamma_u \), where \( \gamma_u \) is the unipotent part of the monodromy \( \gamma \). We recall [5] that, if \( k \) is the weight of the Hodge structures, \( N^{k+1} = 0 \) and the filtration \( \mathcal{N} = (0) \subset W_0 \subset \cdots \subset W_2 = H \) is uniquely characterized by the conditions \( NW_j \subset W_{j-2} \) and \( W_{k+j}/W_{k+j-1} \to W_{k-j}/W_{k-j-1} \) is an isomorphism.

The results announced here concern the monodromy weight filtrations arising in the several variables case. The main statements—Theorems 1 and 2—were conjectured by P. Deligne [2] (cf. Conjecture 1.9.6, as well as Theorem 1.9.2 for the special geometric case). For structures of weight two they are contained in [1].

**THEOREM 1.** Let \( \gamma_1, \ldots, \gamma_l \) be monodromy transformations of a period mapping \( \psi: (\mathbb{A}^*)^l \times (\Delta)^m \to \Gamma \setminus \mathcal{D} \) and \( N_i \) the logarithm of the unipotent part...
of $\gamma_t$. Then all the elements in the open cone of commuting nilpotent endomorphisms

$$\sigma = \left\{ \sum_{i=1}^{l} \lambda_i N^i; \lambda_i \in \mathbb{R}, \lambda_i > 0 \right\}$$

define the same monodromy weight filtration (to be denoted by $W_*(\sigma)$).

According to Schmid's Nilpotent Orbit Theorem [5] there exists a limiting Hodge filtration $F^*$ (depending on $w_1, \ldots, w_m$) such that the lifting $\tilde{\psi}$ may be approximated for $\text{Im } z_j > 0$ by the orbit $\exp(\Sigma \tau_j N_j) \cdot F^*$. Moreover in the case of a single $N$ and as a consequence of the $SL_2$-orbit theorem, $(W_*(N), F^*)$ defines a polarized mixed Hodge structure, i.e. the Hodge filtration $F^*$ defines Hodge structures on the graded quotients $\text{Gr}^j(N) = W_j(N)/W_{j-1}(N)$ which are suitably polarized (cf. Theorem 6.16 in [5] for the precise statement, as well as [6]). When this is combined with Theorem 1 one obtains that every point in the approximating orbit defines a mixed Hodge structure relative to the filtration $W_*(\sigma)$.

A further consequence of Theorem 1 is the existence of a Hodge filtration $F^*_0$ such that $(W_*(\sigma), F^*_0)$ is a mixed Hodge structure split over $\mathbb{R}$ and the orbit $\exp(\Sigma \tau_j N_j) \cdot F^*_0$ lies in $D$ for $\text{Im } z_j > 0$. For a single monodromy transformation these "split" nilpotent orbits correspond to $SL_2$-orbits in the sense of [5]. Thus they could be expected to play a role in extending the $SL_2$-orbit theorem to the case of period mappings of several variables.

The second theorem relates the weight filtration $W_*(\sigma)$ to those associated to the faces of the cone $\sigma$. In order to make this statement precise let us consider two commuting nilpotent transformations $N$ and $N'$. Since $N'$ preserves $W_*(N)$, it induces nilpotent endomorphisms in each of the graded quotients $\text{Gr}_j(N)$. Suppose that the monodromy weight filtrations defined by $N'$ in the various $\text{Gr}_j(N)$'s are all projections of a single filtration $W_*$ of the total space with the property that $N'W_j \subseteq W_{j-2}$ (such a $W_*$ need not exist for an arbitrary commuting pair $N, N'$, but if it does, it is unique [2]). Then, following Deligne, we call such $W_*$ the monodromy weight filtration of the pair $(N', W_*(N))$.

**Theorem 2.** Let $N, N'$ be any two elements in the closure of $\sigma$ not belonging to the same proper face of this cone. Then $W_*(\sigma)$ is the monodromy weight filtration of the pair $(N', W_*(N))$.

We end with a bare bones sketch of the proof of Theorem 1. For elementary reasons there exists a Zariski-open subcone $\sigma'$ of $\sigma$ where the map $N \rightarrow W_*(N)$ takes values in a fixed flag manifold. Then the existence of a Hodge filtration $F^*$ such that $(W_*(N), F^*)$ is a polarized mixed Hodge structure for all $N \in \sigma$ is seen to imply the vanishing of the differential of that map. Hence the filtration $W_*(N)$ must be constant on each connected component of $\sigma'$. 
For any such component $o'_0$, it is now possible to construct a new Hodge filtration $F^*_0$ such that $(W_*(o'_0), F^*_0)$ is a mixed Hodge structure split over $\mathbb{R}$ and such that $\exp zN \cdot F^*_0 \in D$ for $N \in o'_0$ and $\text{Im } z > 0$. This, combined with Lemma 1 in [1], guarantees in turn that $(W_*(N), F^*_0)$ is a mixed Hodge structure even for an $N$ in the closure of $o'_0$ in $\alpha$. The conclusion that $W_*(N) = W_*(o'_0)$ for any such $N$ and, hence, that of Theorem 1, follows from the Proposition below which, together with the reduction to the $\mathbb{R}$-split case, also underlies the proof of Theorem 2.

**Proposition.** Let $(W_*, F^*)$ be a mixed Hodge structure split over $\mathbb{R}$ and $N$ a $(-1, -1)$-morphism of it. Then $(W_*(N), F^*)$ is a mixed Hodge structure if and only if $W_*(N) = W_*$. 

Detailed proofs will appear elsewhere. We wish to thank Pierre Deligne for his generous advice and encouragement during the preparation of this work.

**REFERENCES**


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