SMOOTH BOUNDED STRICTLY AND WEAKLY
PSEUDOCONVEX DOMAINS CANNOT BE BIHOLOMORPHIC

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There is no Riemann mapping theorem in the theory of functions of several
complex variables; nor is there a Riemann nonmapping theorem. In fact, until
recently, it was not known whether it is possible for a smooth bounded strictly
pseudoconvex domain to be biholomorphically equivalent to a smooth bounded
weakly pseudoconvex domain. The paper [1] answers this question in the nega­
tive by proving

THEOREM 1. If $D_2$ is a smooth bounded pseudoconvex domain, and $D_1$
is a smooth bounded domain whose $\bar{\partial}$-Neumann problem satisfies global regularity
estimates, then biholomorphic mappings between $D_1$ and $D_2$ extend smoothly to
the boundary.

In particular, it is known (Kohn [6]) that the $\bar{\partial}$-Neumann problem satisfies
global regularity estimates in smooth bounded strictly pseudoconvex domains.
Hence, a biholomorphic mapping between a smooth bounded strictly pseudocon­
vex domain and a smooth bounded weakly pseudoconvex domain would extend
smoothly to the boundary. Since strict pseudoconvexity is preserved under bi­
holomorphic mappings which extend to be $C^2$ up to the boundary, both domains
must be strictly pseudoconvex.

Other domains for which the $\bar{\partial}$-Neumann problem is known to satisfy glo­
bal regularity estimates include smooth bounded weakly pseudoconvex domains
with real analytic boundaries [7], [2] and certain domains of finite type [7].

The proof of Theorem 1 exploits the transformation rule for the Bergman
projection. If $P_i$ denotes the Bergman orthogonal projection of $L^2(D_i)$ onto its
subspace of holomorphic functions, $i = 1, 2$, and if $f: D_1 \to D_2$ is a biholo­
morphic mapping, then

$$P_1(u \cdot (\phi \circ f)) = u \cdot ((P_2 \phi) \circ f)$$

where $u = \text{Det} [f']$ and $\phi \in L^2(D_2)$. It is possible to construct functions $\phi$ which
vanish to arbitrarily high order on $bD_2$ such that $P_2 \phi \equiv 1$. If $\phi$ is such a function
which vanishes to a high enough order on $bD_2$ to make $u \cdot (\phi \circ f)$ smooth up to
the boundary, then $u$ is the projection of a function which is smooth up to the
boundary. Since global estimates for the $\overline{\partial}$-problem in $D_1$ imply global estimates for $P_1$, we conclude that $u$ is smooth up to the boundary.

A similar argument reveals that $u \cdot (g \circ f)$ is smooth up to the boundary whenever $g$ is a holomorphic function on $D_2$ which is smooth up to the boundary.

The final steps in the proof of Theorem 1 depend on Kohn's theory of the $\overline{\partial}$-Neumann problem with weight functions [8] and a special Sobolev inequality for holomorphic functions: If $g$ and $h$ are holomorphic functions on a smooth bounded domain $D$, then

$$\left| \int_D h \overline{g} \right| \leq C \|h\|_s \|g\|_{-s}$$

where $\|h\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha h\|_{L^2(D)}^2$ is the usual Sobolev $s$-norm and

$$\|g\|_{-s} = \sup_{\phi \in C_0^\infty(D)} \left\| \left( g, \phi \right)_{L^2(D)} \right\|_{s=1}$$

The constant $C$ only depends on $D$ and the positive integer $s$. This inequality is unique to holomorphic functions and must not be confused with the standard generalized Schwarz inequality.

Theorem 1 generalizes and improves several earlier results on biholomorphic mappings due to Henkin [5], Pincuk [9], Fefferman [4], Diederich and Fornaess [3]. The Henkin, Pincuk, Diederich and Fornaess results also apply to proper holomorphic mappings. At the present time, the techniques used in the proof of Theorem 1 do not seem to adapt to the proper mapping case.

REFERENCES

1. S. Bell, Biholomorphic mappings and the $\overline{\partial}$-problem, Ann. of Math. (2) (to appear).

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