FOR $n > 3$ THERE IS ONLY ONE
FINITELY ADDITIVE ROTATIONALLY INVARIANT MEASURE
ON THE $n$-SPHERE DEFINED
ON ALL LEBESGUE MEASURABLE SUBSETS
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The following paragraph is taken from the introduction of Joseph Rosenblatt’s paper [R].

“Let $\beta$ be the ring of Lebesgue measurable sets in the $n$-sphere $S^n$, and let
$\lambda_n$ denote the Lebesgue measure on $\beta$ normalized by $\lambda_n(S^n) = 1$. The classical
characterization by Lebesgue of $\lambda_n$ is that it is the unique positive real-valued
function $\mu$ on $\beta$ which satisfies these three conditions:

(a) $\mu(S^n) = 1$;
(b) $\mu$ is invariant under isometries;
(c) $\mu$ is countably additive.

In 1923 Banach [B] studied the question of Ruziewicz whether $\mu$ is still unique
when (c) is replaced by

(c$_0$) $\mu$ is finitely additive.

Banach gave a negative answer to this question for $S^1$ but for $S^n$, $n \geq 2$, the
question is still unanswered.”

From the body of Rosenblatt’s paper one can extract the implication that
if Lebesgue measure $\lambda_n$ on $S^n$ is not characterized by (a), (b), and (c$_0$) then
there is a net of measurable subsets $(A_\alpha) \subset S^2$ which is asymptotically invariant
and nontrivial, namely $\lim_{\alpha}(\lambda_n(gA_\alpha \Delta A_\alpha)/\lambda_nA_\alpha) = 0$ for all rotations $g$ and so
that $0 < \lambda_n(A_\alpha) \leq c < 1$ (Theorem 1.4 of [R]). Here $A \Delta B = A \cup B - A \cap B$.

The following Proposition will show that such asymptotically invariant nets
on $S^n$ are impossible, $n > 3$.

PROPOSITION. For each $n > 3$ there is a countable subgroup $\Gamma_n$ in the
group $O_{n+1}$ of rotations of $S^n$ satisfying

(i) the action of $\Gamma_n$ on $S^n$ is ergodic,
(ii) the group $\Gamma_n$ satisfies Kazhdan’s property $T$:

There exist a finite subset $\Lambda \subset \Gamma_n$ and an $\varepsilon > 0$, so that for any unitary
representation $\pi$ if $\Gamma$, if there exists a vector $\xi$ in $H_n$ such that $||\xi|| = 1$,
\[ \| \pi(g) \xi - \xi \| \leq \epsilon \quad \forall g \text{ in } \Lambda \text{ then there exists a vector } \xi' \in H_{\pi} \text{ with } \pi(g) \xi' = \xi' \quad \forall g \in \Gamma_n, \text{ and } \xi' \neq 0. \]

**Proof.** For \( n > 3 \) let \( \Gamma_n \) be the group of \((n + 1) \times (n + 1)\) matrices with entries integers \((n + m\sqrt{2})\) of the field \( Q(\sqrt{2}) \) where such matrices preserve the quadratic form

\[ x_0^2 + x_1^2 + \cdots + x_{n-2}^2 - \sqrt{2}x_{n-1}^2 - \sqrt{2}x_n^2. \]

If we conjugate all the matrices of \( \Gamma_n \) by the field automorphism of \( Q(\sqrt{2}) \) we obtain a group of matrices isomorphic to \( \Gamma_n \) preserving the form

\[ x_0^2 + x_1^2 + \cdots + x_{n-2}^2 + \sqrt{2}x_{n-1}^2 + \sqrt{2}x_n^2. \]

So \( \Gamma_n \) is embedded as a subgroup of \( O(n+1) \), the real orthogonal group of the second quadratic form. If \( O(n-1, 2) \) denotes the real orthogonal group of the first quadratic form then the diagonal embedding \( \Gamma_n \rightarrow O(n+1) \times O(n-1, 2) \) is discrete because the diagonal embedding \((n + m\sqrt{2}) \in Q(\sqrt{2}) \rightarrow (n + m\sqrt{2}, n - m\sqrt{2}) \in R \times R \) is discrete. By a basic theorem of arithmetic groups \( \Gamma_n \) has cofinite volume in \( O(n+1) \times O(n-1, 2) \). Since \( O(n+1) \) is compact, \( \Gamma_n \) is discrete with cofinite volume in \( O(n-1, 2) \).

Since \( O(n-1, 2) \) is a simple Lie group of real rank \( \geq 2 \) it has Kazhdan's property (see [K]) which descends by an averaging argument (Theorem 3 of [K]) to the discrete subgroup with cofinite volume \( \Gamma_n \). Thus \( \Gamma_n \) has Kazhdan's property \( T \). This proves (ii).

Now if the topological closure of \( \Gamma_n \subset O(n+1) \) were a proper closed subgroup \( G \) then the complexification \( G_C \) of \( G \) in the complexification \( O(n+1, C) \) of \( O(n+1) \) would define a proper C-algebraic subgroup containing \( \Gamma_n \). But for the conjugate embedding \( \Gamma_n \subset O(n-1, 2) \subset O(n+1, C) \), \( \Gamma_n \) is Zariski dense by Borel's density theorem. This is a contradiction showing \( \Gamma_n \) is topologically dense in \( O(n+1) \).

Since \( \Gamma_n \) is a dense subgroup of isometries ergodicity follows immediately from a consideration of Lebesgue density points. This proves (i).

Combining the Proposition with Rosenblatt's work [R] we have the answer to the Banach-Ruziewicz problem, \( n > 3 \).

**Theorem.** Spherical measure on \( S^n, n > 3 \), is the only finitely additive normalized measure invariant under rotations and defined\(^1\) on all Lebesgue measurable sets.

**Proof.** If not by Rosenblatt [R] there is, as mentioned above, a nontrivial asymptotically invariant net of sets \((A_\alpha) \subset S^2\). Clearly we can extract a \(1\text{In } [R] \) one finds Tarski's observation using paradoxical decompositions that if a finitely additive measure is defined on all Lebesgue measurable sets it must be zero on Lebesgue null sets.
countable subsequence \((A_j) \subset S^2\) which is asymptotically invariant for the countable subgroup \(\Gamma_n \subset O_n+1\) constructed in the Proposition. Namely, \(0 < \lambda_n(A_j) \leq c < 1\), and for all \(g \in \Gamma_n\) \(\lim (\lambda_n(gA_j \Delta A_j)/\lambda_n(A_j)) = 0\).

Now convert the characteristic \(\chi\) of \(A_j\) into functions of integral zero by forming \(f_j = (\chi A_j/\sqrt{\lambda_n(A_j)}) - \sqrt{\lambda_n(A_j)}\) and then \(F_j = f_j/\|f_j\|_2\). Then \(\int f_j d\lambda_n = 0\) because \(\int f d\lambda_n = 0\), and \(\|F_j\|_2 = 1\). Also

\[
\|f_j \circ g - f_j\|_2^2 = 2(1 - \lambda_n(g^{-1}A_j \cap A_j)/\lambda_n(A_j)).
\]

Since \(\|f\|_2^2 = \lambda_n(S^2 \mid A_j)\), it is bounded away from zero by the nontriviality of \((A_j)\). Thus \(\lim \|F_j \circ g - F_j\|_2 = 0\) for all \(g \in \Gamma_n\) and \(\|F_j\|_2 = 1\). (Compare [R, Lemma 3.1].)

If we apply property \(T\) for \(\Gamma_n\) for the representation of \(\Gamma_n\) on the space \(H\) of square integrable functions \(f\) on \(S^n\) of integral zero we obtain from the existence of the vectors \(F_j\) of \(H\), the existence of an element in \(H\) of norm 1 which is \(\Gamma_n\) invariant. This contradicts ergodicity of \(\Gamma_n\). Thus there is no such net of asymptotically invariant sets, and the Theorem is proved.

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REFERENCES


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