RESEARCH ANNOUNCEMENTS

THE TETRAGONAL CONSTRUCTION

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1. Preliminaries. Let $C$ be a nonsingular curve of genus $g$, and $\pi: \widetilde{C} \rightarrow C$ an unramified double cover. The Prym variety $P(C, \widetilde{C})$ is by definition $\ker^0(Nm)$, where $Nm: J(\widetilde{C}) \rightarrow J(C)$ is the norm map, and $\ker^0$ is the connected component of 0 in the kernel. By [M] this is a $(g-1)$-dimensional, principally polarized abelian variety. Let $A_g$, $M_g$, $R_g$ denote, respectively, the moduli spaces of $g$-dimensional principally polarized abelian varieties, curves of genus $g$, and pairs $(C, \widetilde{C})$ as above. ($R_g$ is a $(2^{2g} - 1)$-sheeted cover of $M_g$.) The Prym map is the morphism

$$P = P_g: R_g \rightarrow A_{g-1}, \quad (C, \widetilde{C}) \mapsto P(C, \widetilde{C}).$$

It is analogous to the Jacobi map $J = J_g: M_g \rightarrow A_g$ sending a curve to its Jacobian. The main reason for studying $P$ is that its image in $R_{g-1}$ is larger than that of $J$, hence it allows us to handle geometrically a wider class of abelian varieties than just Jacobians. For instance, $P_g$ is dominant for $g \leq 6$ [W] while $J_g$ is only dominant for $g \leq 3$.

The purpose of this announcement is to describe the fibers of $P$ in the various genera. Our main tool for this is a simple-minded construction which we describe in some detail in paragraph 6. Let us use "$n$-gonal" (trigonal, tetragonal, etc.) to describe a pair $(C, f)$ where $f: C \rightarrow \mathbb{P}^1$ is a branched cover of degree $n$ (3, 4 respectively). Briefly, our construction takes the data $(C, \widetilde{C}, f)$ where $(C, \widetilde{C}) \in R_g$ and $(C, f)$ is tetragonal, and returns two new sets of data, $(C_0, \widetilde{C}_0, f_0)$ and $(C_1, \widetilde{C}_1, f_1)$, of the same type. This procedure is symmetric: starting with $(C_0, \widetilde{C}_0, f_0)$ we end up with $(C, \widetilde{C}, f)$ and $(C_1, \widetilde{C}_1, f_1)$. It is useful due to the following observation.

**Proposition 1.1.** The tetragonal construction commutes with the Prym map:

$$P(C, \widetilde{C}) \approx P(C_0, \widetilde{C}_0) \approx P(C_1, \widetilde{C}_1).$$

Received by the editors September 3, 1980,
1980 Mathematics Subject Classification. Primary 14H15, 14H30, 14K10, 32G20, 14H40.

1Research partially supported by NSF Grant MCS #77-03876.
Remark 1.2. A similar construction was studied by Recillas [R], [DS, III]. He starts with a tetragonal pair \((C, f)\) and produces a triplet \((X, \tilde{X}, g)\) where \((X, g)\) is trigonal and \(P(X, \tilde{X}) \approx J(C)\). This becomes the special case of our construction where \(\tilde{C}\) is taken to be the split double cover of \(C\). The resulting \(C_0, C_1\) are then isomorphic to \(X\) with a \(P^1\) attached (in two different ways) and

\[ P(C_i, \tilde{C}_i) \approx P(X, \tilde{X}) \approx J(C) \approx P(C, \tilde{C}). \]

2. Genus 6. In [DS] the map \(P: \mathcal{R}_6 \to \mathcal{A}_5\) was studied at length. The main result was that this map is generically finite, of degree 27.

Theorem 2.1. The fibers of \(P: \mathcal{R}_6 \to \mathcal{A}_5\) have a structure equivalent to the intersection-configuration of the 27 lines on a cubic surface.

An equivalent formulation is

Corollary 2.2. The Galois group of the field extension \(K(\mathcal{A}_5) \subset K(\mathcal{R}_6)\) is the Weyl group \(W(E_6)\). (Compare [Ma, Theorem 23.9]).

The theorem limits severely the possible degenerations in a fiber of \(P\).

For instance

Corollary 2.3. The ramification locus (in \(\mathcal{R}_6\)) is mapped six-to-one to the branch locus (in \(\mathcal{A}_5\)).

Proof. A line on a cubic surface \(S\) counts twice if and only if it passes through a double point of \(S\). Through such a point there are six lines. \(\square\)

The proof of the theorem depends on the existence of 5 tetragonal maps, \(f_i\) \((1 \leq i \leq 5)\) on a generic curve \(C\) of genus 6. To each triplet \((C, \tilde{C}, f_i)\) the tetragonal construction associates two others; the ten resulting points of \(P^{-1}P(C, \tilde{C})\) are the ones "incident" to \((C, \tilde{C})\).

The same method allows us to recover the main result of [DS] rather painlessly: we show that starting with \((C, \tilde{C}) \in \mathcal{R}_6\), choosing a tetragonal \(f\), applying the tetragonal construction to get \((C_0, \tilde{C}_0, f_0)\), changing the tetragonal \(f_0\) to an \(f'\) and repeating the process indefinitely, leads to precisely 27 distinct objects: to the original \((C, \tilde{C})\) are added ten after the first cycle, and only sixteen more after the second cycle. (I.e. each of the five first-generation pairs yields the same set of sixteen second-generation objects!) Therefore \(\text{deg}(P)\) is a multiple of 27. This possible multiplicity is eliminated by checking a degenerate case, where \(C\) is a double cover (branched) of an elliptic curve ("elliptic hyper-elliptic").

3. Genus 5. The map \(P_5: \mathcal{R}_5 \to \mathcal{A}_4\) turns out, surprisingly, to be more intricate than its higher-genus cousin \(P_6\), and until now has eluded description.
By dimension count, the generic fiber is 2 dimensional; we show that in fact it is a double cover of a Fano surface.

**Theorem 3.1.** There is a birational isomorphism $\kappa: A_4 \to C$ where $C$ is a parameter-space for pairs $(X, \mu)$ consisting of (the isomorphism class of) a cubic threefold $X$ together with an "even" point of order two in its intermediate Jacobian.

**Proposition 3.2.** There is a natural involution $\lambda: R_5 \to R_5$ such that $\lambda(C, \tilde{C})$ is related to $(C, \tilde{C})$ by a succession of two tetragonal constructions; hence $P \circ \lambda = P$.

**Theorem 3.3.** For generic $A \in A_4$, the quotient $P^{-1}(A)/\lambda$ is isomorphic to $F(\kappa(A))$, the Fano surface of lines on the cubic threefold $\kappa(A)$.

The proofs seem to depend heavily on the results for genus 6 and their various specializations. As a corollary, we have an explicit parametrization of the family of (rational equivalence classes of) effective symmetric representatives of the class $[\theta] / 3$ in $H_2(A, \mathbb{Z})$. This is twice the class of a curve in its Jacobian, and the smallest class which is effective on generic $A$.

**4. Prym-Torelli.** For $g \leq 4$ the analysis of $P_g$ is fairly easy. It can be done using nothing but Recillas' trigonal construction (1.2), since any $A \in A_{g-1}$ is Jacobian of a tetragonal curve. In the remaining cases $g \geq 7$, $P_g$ "ought" to be injective by dimension count. After some inconclusive work of Tjurin [T], counterexamples to this expected Prym-Torelli theorem were exhibited by Beauville [B2] for $g < 10$, using Recillas' construction applied to curves which are tetragonal in two distinct ways. Using the tetragonal construction we exhibit counterexamples for all $g$. Without much justification we make the following

**Conjecture 4.1.** If $P(C, \tilde{C}) \approx P(C', \tilde{C}')$ then $(C', \tilde{C}')$ is obtained from $(C, \tilde{C})$ by successive applications of the tetragonal construction. In particular, $C$ and $C'$ are tetragonal curves.

**5. Andreotti-Mayer varieties.** In [AM], Andreotti and Mayer studied the Schottky problem of characterizing Jacobians among abelian varieties. Call $A \in A_g$ an $A-M$ variety if its theta divisor $\theta$ has a $(g - 4)$-dimensional singular locus, and let $N_g \subset A_g$ be the closure of the locus of $A-M$ varieties. The main results of [AM] are that $N_g$ can be explicitly described by equations, and that $\mathcal{J}(M_g)$ is an irreducible component of $N_g$. Perhaps the most spectacular application of Prym theory was Beauville's refinement of their results [B1]. He obtained a complete (and lengthy) list of all possible components of $P^{-1}(N_g)$, hence, in principle, a description of $N_4, N_5$ (since $P_5, P_6$ are surjective, when appropriately
compactified). In particular, he showed that $N_4$ has only one irreducible component other than $J(M_4)$.

Using the tetragonal construction, some remarkable coincidences appear in Beauville's list. In fact

**Theorem 5.1.** (1) $N_4$ consists of $J_4$ and another nine-dimensional irreducible component [B1].

(2) $N_5$ consists of $J_5$ and four irreducible, nine-dimensional loci; three of these parametrize Pryms of elliptic-hyperelliptic curves, and the fourth consists of certain abelian varieties isogenous to a product with an elliptic curve.

(3) For $g \geq 6$, $N_g \cap \overline{\mathcal{P}(R_{g+1})}$ consists of $J_g$, 2 components of Pryms of elliptic-hyperelliptic curves (each $(2g - 1)$ dimensional) and $[(g - 2)/2]$ components of Pryms of reducible curves $C = C_1 \cup C_2$, $\#(C_1 \cap C_2) = 4$ (each $(3g - 4)$ dimensional).

**Corollary 5.2.** Any $(C, \widetilde{C}) \in \mathcal{P}^{-1}(N_g)$ is either tetragonal (or a degeneration of tetragons) or reducible. The modified Prym-Torelli Conjecture 4.1 holds over $N_g$.

**Conjecture 5.3.** $N_g \subset \overline{\mathcal{P}(R_{g+1})}$, hence $N_g$ consists only of the components listed above.

The proof might imitate Andreotti's proof of Torelli's theorem and resurrect Tjurin's work [T]: Given $A \in N_g$, there should be some explicit geometric construction yielding a family of doubly covered tetragonal (or reducible) curves, whose Prym is $A$.

**Corollary 5.4.** For any canonical curve $C \subset \mathbb{P}^{g-1}$, the system of quadrics containing $C$ is spanned by quadrics of rank 4.

**Proof.** A refinement of [AM] shows that the truth of the corollary for a given $C$ depends only on the structure of $N_g$ near $J(C)$; in particular the corollary holds if $J_g$ is the only component of $N_g$ containing $J(C)$. By Conjecture 5.3 and Theorem 5.1, this holds for all $C$ except for hyperelliptics and elliptic-hyperelliptics. A special argument works for these.

6. The construction. We sketch the tetragonal construction. Start with an unramified double cover $\pi: \widetilde{C} \to C$ and tetragonal map $f: C \to \mathbb{P}^1$. Let $f_*(\pi): f_*(\widetilde{C}) \to \mathbb{P}^1$ be the "pushforward" of $\pi: \widetilde{C} \to C$ via $f$. This is a $(16 = 2^4)$-sheeted branched cover. Over $p \in \mathbb{P}^1$, its 16 points correspond to the 16 ways of lifting the quadruple $f^{-1}(p) \subset C$ to a quadruple in $\widetilde{C}$. This suggests a convenient way of realizing $f_*(\widetilde{C})$ as a curve in Pic$^{(4)}(\widetilde{C})$, the Picard variety of line bundles of degree 4: $f_*(\widetilde{C})$ is the subvariety parametrizing those effective divisors in $\widetilde{C}$ whose norm
(under $\pi$, $\widetilde{C} \to C$) is in the 1-dimensional linear series determined on $C$ by $f$.

Note that on the curve $f_*\widetilde{(C)}$ there is a natural involution $\tau$: $f_*\widetilde{(C)} \to f_*\widetilde{(C)}$. $\tau$ sends a lifting of $f^{-1}(p)$ to the complementary lifting, obtained by interchanging the sheets of $\pi$: $\widetilde{C} \to C$. (This is induced by the automorphism of $\text{Pic}^4(\widetilde{C})$ sending a line bundle $L$ to $L^{-1} \otimes (f \circ \pi)^* \mathcal{O}_{\mathbb{P}^1}(1)$.) Let $\widetilde{C}$ be the quotient $f_*\widetilde{(C)}/\tau$, an 8-sheeted cover of $\mathbb{P}^1$.

**Lemma.** $\widetilde{C}$ is reducible: $\widetilde{C} = C_0 \cup C_1$, each $C_i$ is a 4-sheeted branched cover of $\mathbb{P}^1$. Correspondingly, $f_*\widetilde{(C)} = \widetilde{C}_0 \cup \widetilde{C}_1$, where $\widetilde{C}_i$ is acted upon by $\tau$ with quotient $C_i$.

**Proof.** Define an equivalence relation $\sim$ on $f_*\widetilde{(C)}$: $D_1 \sim D_2$ if $f_*(\pi)(D_1) = f_*(\pi)(D_2)$ and $D_1, D_2$ have an even number of points (0, 2 or 4) in common. The quotient $f_*\widetilde{(C)}/\sim$ is a 2-sheeted branched cover of $\mathbb{P}^1$. Clearly it can be branched only where $f: C \to \mathbb{P}^1$ is; but a simple monodromy check shows that at such a point $f_*\widetilde{(C)}/\sim$ is locally reducible. (I.e. in going around a branch point, an even number of points of $\widetilde{C}$ are exchanged.) Hence the normalization of $f_*\widetilde{(C)}/\sim$ is nowhere ramified over $\mathbb{P}^1$, hence consists of two disjoint copies, so $f_*\widetilde{(C)}$ itself is reducible. Finally, $\tau$ acts on each component separately since it changes an even number (all 4) of the points. Q.E.D.

**Note.** Identifying $\text{Pic}^4(\widetilde{C}) \approx \text{Jac}(\widetilde{C})$, we have that $f_*\widetilde{(C)}$ is contained in the kernel of the norm-homomorphism, which $[M]$ consists of two copies of the Prym variety; $\widetilde{C}_i$ are the intersections with these two components.

**Acknowledgements.** My warmest thanks to A. Beauville, C. H. Clemens, R. Smith, and R. Varley for numerous conversations, ideas and much encouragement.

**References**


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