THE MULTIPLICITY OF EIGENVALUES

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There are many examples of first order $n \times n$ systems of partial differential equations in 2 space variables with real coefficients which are strictly hyperbolic; that is, they have simple characteristics. In this note we show that in 3 space variables there are no strictly hyperbolic systems if $n \equiv 2(4)$. Multiple characteristics of course influence the propagation of singularities. For a different context see Appendix 10 of [2].

$\mathcal{M}$ denotes the set of all real $n \times n$ matrices with real eigenvalues. We call such a matrix nondegenerate if it has $n$ distinct real eigenvalues.

**THEOREM.** Let $A, B, C$ be three matrices such that all linear combinations

\[ \alpha A + \beta B + \gamma C, \]

$\alpha, \beta, \gamma$ real, belong to $\mathcal{M}$. If $n \equiv 2 \pmod{4}$, then there exists $\alpha, \beta, \gamma$ real, $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ such that (1) is degenerate.

**REMARK 1.** The theorem applies in particular to $A, B, C$ real symmetric.

**REMARK 2.** The theorem shows that first order hyperbolic systems in three space variables of the indicated order always have some multiple characteristics.

**PROOF.** Denote by $\mathcal{N}$ the set of nondegenerate matrices in $\mathcal{M}$. The normalized eigenvectors $u$ of $\mathcal{N}$ is $\mathcal{N}$,

\[ \mathcal{N} u_j = \lambda_j u_j, \quad |u_j| = 1, \quad j = 1, \ldots, n, \]

are determined up to a factor $\pm 1$.

Let $\mathcal{N}(\theta), 0 < \theta < 2\pi$, be a closed curve in $\mathcal{N}$. If we fix $u_j(0)$, then $u_j(\theta)$ can be determined uniquely by requiring continuous dependence on $\theta$. Since $\mathcal{N}(2\pi) = \mathcal{N}(0)$,

\[ u_j(2\pi) = \tau_j u_j(0), \quad \tau_j = \pm 1. \]

Clearly

(i) Each $\tau_j$ is a homotopy invariant of the closed curve.

(ii) Each $\tau_j = 1$ when $\mathcal{N}(\theta)$ is constant.

Suppose now that the theorem is false; then

\[ \mathcal{N}(\theta) = \cos \theta A + \sin \theta B \]

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is a closed curve in $N$. Note that $N(\pi) = -N(0)$; this shows that
\[
\begin{align*}
\lambda_j(\pi) &= -\lambda_{n-j+1}(0) \quad \text{and} \\
u_j(\pi) &= \rho_j u_{n-j+1}(0), \quad \rho_j = \pm 1.
\end{align*}
\]

Since the ordered basis $\{u_1(\theta), \ldots, u_n(\theta)\}$ is deformed continuously, it retains its orientation. Thus the ordered bases
\[
\{u_1(0), \ldots, u_n(0)\} \quad \text{and} \quad \{\rho_1 u_n(0), \ldots, \rho_n u_1(0)\}
\]
have the same orientation. For $n \equiv 2 \pmod{4}$, reversing the order reverses the orientation of an ordered base; this proves that
\[
\prod_{j=1}^n \rho_j = -1.
\]

This implies that there is a value of $k$ for which
\[
\rho_k \rho_{n-k+1} = -1.
\]

Next we observe that $N(\theta + \pi) = -N(\theta)$; it follows from this that $\lambda_j(\theta + \pi) = -\lambda_{n-j+1}(\theta)$ and by (4) that
\[
u_j(2\pi) = \rho_{n-j+1} u_{n-j+1}(\pi).
\]
Combining this with (4) we get that $\tau_j = \rho_j \rho_{n-j+1}$. By (5), $\tau_k = -1$; this shows that the curve (3) is not homotopic to a point.

Suppose that all matrices of form (1), $\alpha^2 + \beta^2 + \gamma^2 = 1$, belonged to $N$. Then since the sphere is simply connected the curve (4) could be contracted to a point, contradicting $\tau_k = -1$.

See [1] for related matters.

**ADDED IN PROOF.** S. Friedland, J. Robbin and J. Sylvester have proved the theorem for all $n \equiv \pm 2, \pm 3, \pm 4 \pmod{8}$, and have shown it false for $n = 0, \pm 1 \pmod{8}$. They have further results involving linear combinations of more than 3 matrices.

**REFERENCES**


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