A NOTE ON A PROBLEM OF SAFF AND VARGA
CONCERNING THE DEGREE
OF COMPLEX RATIONAL APPROXIMATION
TO REAL VALUED FUNCTIONS

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Saff and Varga [1977, 1978] discovered the surprising fact that given a
real valued function on $[-1, 1]$ one can sometimes obtain a better rational ap­
proximation to $f$ of a given type by allowing complex coefficients than by re­
stricting the coefficients to be real. In this note we point out a connection
between this result and the “Trefethen effect” (Trefethen [1981a, 1981b]) of near
circular error curves for best approximation on the unit disc, which as Trefethen
has shown is in turn closely related to the Carathéodory-Fejér Theorem and its
generalisation to meromorphic approximation due to Takagi.

We consider a compact simply connected set $\Omega$ in the complex plane
which is symmetric about the real line (i.e. $z \in \Omega$ iff $\overline{z} \in \Omega$). We assume further
that the complement of $\Omega$ is simply connected in the extended plane and that
$\Omega$ is a Faber domain (see e.g. Gaier [1980]). Finally if $\psi$ denotes the conformal
mapping of the complement of the unit disc $D$ onto the complement of $\Omega$ such
that $\psi(\infty) = \infty$, $\lim_{w \to \infty} \psi(w)/w$ finite real and positive, we assume for simpli­
city that $\psi$ can be extended continuously to the boundary of the disc. All
these properties are, of course, satisfied for the main case of interest here, $\Omega =
I = [-1, 1]$.

$\mathfrak{A}(\Omega)$ will denote the set of functions continuous on $\Omega$ and analytic at
interior points, and $\mathfrak{A}^R(\Omega)$ the subspace of functions satisfying $f(\overline{z}) = \overline{f(z)}$.
$\mathfrak{L}: \mathfrak{A}(D) \to \mathfrak{A}(\Omega)$ is the Faber transform (Gaier [1980]); $E^C_{mn}(f; \Omega)$ the error
of best (Chebyshev) approximation to $f \in \mathfrak{A}^R(\Omega)$ from $R^R_{mn}(\Omega)$, the set of
type $(m, n)$ rational functions with no poles in $\Omega$, and $E^R_{mn}(f; \Omega)$ the correspond­
ing error of best approximation from the subset of rational approximations with
real coefficients.

Saff and Varga [1977, 1978] observed that it is possible to find $f \in \mathfrak{A}^R(I)$
for which $E^C_{nn}(f; I) < E^R_{nn}(f; I)$ (e.g. $f(x) = x^2$, $n = 1$). Some authors attacked
the case $n = 1$ by largely geometric arguments and showed among other things

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that for certain classes of functions $E_{11}^R(f; I)/E_{11}^C(f; I) \geq \frac{1}{2}$ (Ruttan [1977]; Bennet, Rudnick and Vaaler, [1979]). We consider more generally $m \geq n - 1$ and obtain a lower bound for $E_{mn}^C(f; \Omega)/E_{mn}^R(f; \Omega)$ which, if $m \geq n$ and $\Omega$ is convex, is for many functions very close to $\frac{1}{2}$.

We introduce the space $\mathcal{R}_{mn}(D)$ which may be described as those functions of the form

$$r(z) = \frac{\sum_{j=-\infty}^{m} a_j z^j}{\sum_{j=0}^{n} b_j z^j}$$

where the denominator has no zeros in $D$ and the numerator represents a function bounded on $S = \{z \mid |z| = 1\}$ and analytic on the complement of $D$. Define $E_{mn}^R(f; S)$ and $E_{mn}^C(f, S)$ for $f \in \mathcal{R}(D)$ in the obvious way.

We now list some simple relationships which are either obvious or straightforward consequences of results given in Trefethen [1981b], Gutknecht [1981] or Ellacott [1981]. Here $g \in \mathcal{R}(D)$.

(i) $E_{mn}^R(g; S) = E_{mn}^C(g; S)$. From now on we drop the superscripts $R$ and $C$ when referring to this error,

(ii) $E_{mn}^R(g; S) \leq E_{mn}^C(g; D) \leq E_{mn}^R(g; D)$ and for many "reasonable" functions $g$ the ratio $\rho(g) = E_{mn}^R(g; D)/E_{mn}^C(g; S)/E_{mn}^R(g; D)$ is very close to 1.

In particular equality holds if $g$ is a polynomial of degree $m + 1$.

(iv) If $m > n - 1, E_{mn}^R(f; S) \leq E_{mn}^C(f; \Omega) \leq E_{mn}^R(f; \Omega)$. For convex $\Omega, \|\mathcal{I}\| \leq 5$ and $\|\mathcal{I}_0\| \leq 2$.

Unfortunately it is not entirely clear at present what a "reasonable" function in the sense of (iii) is but it appears to have some connection with the regularity of the Taylor coefficients. (Trefethen [1981b], has some asymptotic results and some numerical computations for $e^z$, and further numerical results are given in Ellacott [1981]). Not every $f \in \mathcal{R}(\Omega)$ is of the form $\mathcal{I}(g), g \in \mathcal{R}(D)$, but for any function which can be so expressed (e.g. any polynomial or any function with a uniformly and absolutely convergent Faber series) and for $m \geq n - 1$ (iv) and (v) yield immediately

$$\frac{E_{mn}^C(f; \Omega)}{E_{mn}^R(f; \Omega)} \geq \frac{\rho(\mathcal{I}^{-1}(f))}{T}.$$
If $f$ is a polynomial of degree $m + 1$ we have

\[(2) \ E^C_{mn}(f; I)/E^R_{mn}(f; I) \geq 1/2, \ m \geq n,\]

and for "reasonable" functions $f$ (i.e. functions for which $\mathbb{R}^{-1}(f)$ is reasonable in the sense of (iii)), we would not expect the lower bounds given by (1a) and (1b) to be much less than $1/5$ or $1/2$ respectively.

Similar considerations hold for the problem of approximation by the real parts of rational functions as considered by Wulbert [1978].

We conclude with three questions suggested by these remarks. Firstly, and most obviously, is $1/2$ actually a lower bound for $E^C_{mn}(f; I)/E^R_{mn}(f; I)$ for $f \in \mathbb{R}^R(I)$ and $m \geq n$, and, if so, is it sharp?

The second question is related: Are there any functions in $\mathbb{R}^R(D)$ for which $E^C_{mn}(f; D) < E^R_{mn}(f; D)$? If so, they are likely to be more difficult to find than for $I$; (2) shows that $f(z) = z^2$ with $m = n = 1$ will not do. The third question is more general: Certain asymptotic results are known about the behaviour of Faber polynomials as the degree $\to \infty$ (Pommerenke [1964, 1967]). Can these be applied to discuss the asymptotic behaviour of $E^C_{mn}(f; \Omega)/E^R_{mn}(f; \Omega)$?

REFERENCES