FINITE LINEAR GROUPS WHOSE RING OF INVARIANTS IS A COMPLETE INTERSECTION

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ABSTRACT. The celebrated Shephard-Todd-Chevalley theorem says that for a finite linear group $G$ operating on the $n$-dimensional complex vector space the ring $R$ of invariant polynomials is a polynomial ring if and only if $G$ is generated by pseudoreflections ($g \in G$ is a pseudoreflection if $\text{rank}(g - I) = 1$). In this note we give a simple topological proof of the following statement:

If $R$ has $m$ generators such that their ideal of relations is generated by $m - n + s$ elements, then $G$ is generated by those $g \in G$ such that $\text{rank}(g - I) \leq s + 2$.

In the case $s = 0$ this gives a necessary condition for $R$ to be a complete intersection. Our argument also gives a new simple proof of the "only if" part of the Shephard-Todd-Chevalley theorem in the case of an arbitrary ground field.

Let $k$ be a field and let $G$ be a finite subgroup of $GL(n, k)$. The group $G$ acts naturally on the polynomial ring $S = k[x_1, \ldots, x_n]$ and we put $R = S^G$ to be the invariant subring of $G$. We say that $R$ is a polynomial ring if $R$ is generated by $n$ (algebraically independent) elements, and that $R$ is a complete intersection if $R$ is isomorphic to $k[y_1, \ldots, y_{n+r}]/I$, where $J$ is an ideal generated by $r (= \text{emb dim } R - \text{dim } R)$ elements. In this paper we prove the following

**Theorem A.** If $R$ is a complete intersection, then $G$ is generated by the set $\{g \in G| \text{rank}(g - I) \leq 2\}$ (where $I$ is the identity matrix).

The proof is based on two simple topological lemmas. We can assume that the ground field $k$ is algebraically closed.

Let $f: \text{Spec}(S) \rightarrow \text{Spec}(R)$ be the quotient morphism. Let $X'$ and $Y'$ be the henselisations of $\text{Spec}(S)$ at $0$ and of $\text{Spec}(R)$ at $f(0)$ respectively and $f': X' \rightarrow Y'$ the associated morphism. Then the action of $G$ on $\text{Spec}(S)$ lifts to $X'$ and $f'$ is the quotient morphism. We use henselisations in order to deal with simply connected (i.e. without nontrivial étale coverings) schemes $X'$ and $Y'$. If $\text{char } k = 0$, then $\text{Spec}(S)$ and $\text{Spec}(R)$ are simply connected and the henselisation is not necessary.

**Lemma 1.** Let $Y'$ be a simply connected scheme, $Z$ a closed subscheme and $Y = Y' - Z$. If $Y'$ is a complete intersection and $\text{codim } Z \geq 3$, then $Y$ is simply connected.

**Proof.** The proof follows from [2, X, 3.3 and 3.4].

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Remark 1. The conclusion of Lemma 1 holds if instead of \( Y \) to be a complete intersection and \( \text{codim } Z \geq 3 \), we require that \( Y \) is regular and \( \text{codim } Z > 2 \).

Lemma 2 (Vinberg). Let \( X \) be an integral scheme and \( G \) a finite subgroup of \( \text{Aut}_k(X) \). Let \( Y = X/G \) and \( f : X \to Y \) be the quotient morphism. For a closed point \( x \) of \( X \) let \( G_x \) denote the stabilizer of \( x \). If \( Y \) is simply connected, then \( G \) is generated by all \( G_x \)'s.

Proof. Let \( H \) be the subgroup of \( G \) generated by all \( G_x \)'s; it is a normal subgroup. Then for the action of \( G/H \) on \( X/H \) any \( g \neq e \) has no closed fixed points and by [1, I, 10.11], the morphism \( X/H \to Y \) is an étale covering. By our assumption, we have \( G = H \).

Proof of Theorem A. For \( g \in G \) let \( L_g \) denote the subscheme of fixed points of \( g \) on \( X' \). Let \( L \) be the union of all \( L_g \)'s with \( \text{codim } L_g \geq 3 \), and put \( X = X' - L \), \( Z = f'(L) \) and \( Y = Y' - Z \). Note that \( Y' \) is a complete intersection since \( \text{Spec}(R) \) is, and \( Z \) is a closed subscheme in \( Y' \) of codimension \( \geq 3 \). Furthermore, \( X \) is an integral scheme with the induced \( G \)-action, \( Y = X/G \), and \( Y \) is simply connected by Lemma 1. Hence, by Lemma 2, \( G \) is generated by all \( G_x \)'s, \( x \in X \). But by the definition of \( X \), \( g \in G_x \) for some \( x \in X \) if and only if \( \text{codim } L_g \leq 2 \) or, equivalently, \( \text{rank}(g - I) \leq 2 \).

Remark 2. \( R \) is a complete intersection for any \( G \subset GL(2, \mathbb{C}) \) (F. Klein). It is not difficult to construct an example of a finite group \( G \subset SL(3, \mathbb{C}) \) generated by two matrices \( A_1 \) and \( A_2 \), such that \( \text{rank}(A_i - I) = 2 \), \( i = 1, 2 \), but \( R \) is not a complete intersection [7].

Remark 3. Our argument together with Remark 1 gives a short topological proof of the "only if" part of the Shephard-Todd-Chevalley theorem [3, 5] over any ground field \( k \): If \( R \) is a polynomial ring, then \( G \) is generated by pseudoreflections. It is not difficult to show that, furthermore, \( G_x \) is generated by pseudoreflections for any \( x \). The first author takes this opportunity to suggest the following risky conjecture: Conversely, if \( G_x \) is generated by pseudoreflections for any \( x \), then \( R \) is a polynomial ring.

If the ground field is the field \( \mathbb{C} \) of complex numbers, the topology of \( \text{Spec}(R) \) is better known and we can prove the following more general theorem.

Theorem B. Let \( G \) be a finite subgroup of \( GL(n, \mathbb{C}) \) and \( S = \mathbb{C}[x_1, \ldots, x_n] \). If \( R = S^G \) has \( m \) generators such that their ideal of relations is generated by \( m - n + s \) elements, then \( G \) is generated by those \( g \in G \) such that \( \text{rank}(g - I) \leq s + 2 \).
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PROOF. We set $X' = \text{Spec}(S)$, $Y' = \text{Spec}(R)$. By the same argument as above, we have only to prove that $Y = Y' - Z$ is simply connected if $\text{codim } Z \leq s + 3$, under our assumption. The corresponding generalisation of Lemma 1 in the complex case has been recently proved by Goresky and Macpherson [4].

REMARK 4. We do not know whether Theorem B is true for an arbitrary ground field.

Note, finally, that we can strengthen Theorem A (and in a similar way, Theorem B) as follows (cf. Remark 3).

THEOREM C. If $R$ is a complete intersection, then each $G_x$ is generated by \( \{ g \in G_x | \text{rank}(g - I) \leq 2 \} \).

PROOF. Let $X = \text{Spec}(S)$, $Y = X/G_x$ and denote by $\pi: X \to Y$ the quotient morphism. Then the morphism $Y \to X/G$ is étale at $\pi(x)$ by [1, 1, 10.11]. Hence the local ring at $\pi(x) \in Y$ is a complete intersection, and we can apply Theorem A.

REMARK 5. The converse of Theorem C is false (cf. Remark 2).

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