Quillen \cite{7} defines an algebraic K-functor from the category of associative rings to that of positively graded abelian groups, with $K_i(R) = \Pi_i(BGLR^+)$ for $i \geq 1$. $K_1$ and $K_2$ correspond respectively to the 'classical' Bass and Milnor definitions. The $K$-images of finite fields and their algebraic closures were computed by Quillen in \cite{8}. Since then, there has been only a handful of complete calculations of any of the higher $K$-groups ($K_i$ for $i > 2$). Lee and Szczarba \cite{4} showed that the Karoubi subgroup $\mathbb{Z}/48$ of $K_3(\mathbb{Z})$ was the full group. Evens and Friedlander \cite{3} computed $K_i(\mathbb{F}_p[t]/(t^2))$ for $i < 5$ and prime $p$ greater than 3. Snaith in \cite{1} and, with Lluis, in \cite{5}, fully determined $K_3(\mathbb{F}_{p^m}[t]/(t^2))$ for $m \geq 1$ and prime $p$ other than 3.

This note summarizes computations of the groups $K_3(\mathbb{Z}/n)$, and $K_4(\mathbb{Z}/p^k)$ for $k > 1$ and prime $p > 3$. These complete the recent partial results on $K_3(\mathbb{Z}/4)$ by Snaith and on $K_3(\mathbb{Z}/9)$ by Lluis, and extend the work of Evens and Friedlander. The theorem stated below is consistent with the Karoubi conjecture that for odd primes, $BGLZ/p^{k+1}$ is the homotopy fibre of the difference of Adams operations, $\Psi p^k - \Psi p^{k-1}$. However, Priddy \cite{6} has disproved the conjecture in the cases $p > 3$ and $k = 2$.

I am most grateful to Victor Snaith for his supervision of the thesis in which these results originally appeared. Details of the proofs can also be found in \cite{1}.

**THEOREM.** Take $k > 1$ and $0 < i < 2$.

(a) $K_{2i-1}(\mathbb{Z}/2^k) = \mathbb{Z}/2^i \oplus \mathbb{Z}/2^i(2^i-2) \oplus \mathbb{Z}/(2^i-1)$. $K_{2i-1}(\mathbb{Z}/p^k) = \mathbb{Z}/p^{i(k-1)} \oplus \mathbb{Z}/(p^i-1)$ if $p$ is an odd prime. For all primes, the map

$K_{2i-1}(\mathbb{Z}/p^{k+1}) \to K_{2i-1}(\mathbb{Z}/p^k)$

induced by reduction is the obvious surjection.

(b) For prime $p > 3$, $K_2(\mathbb{Z}/p^k) = 0$. $K_2(\mathbb{Z}/3^k) = 0$. $K_2(\mathbb{Z}/2^k) = \mathbb{Z}/2$.

$K_1$ is due to Bass, $K_2$ to Milnor, Dennis Stein.
$K_3(Z/p^k)$ is isomorphic to $H^4(StZ/p^k; Z)$ where the special linear group $SLZ/p^k$ coincides with $StZ/p^k$ modulo $K_3(Z/p^k)$. For odd primes, $K_4(Z/p^k)$ is recovered from the homology of $SLZ/p^k$ using the Serre spectral sequence related to the natural inclusion $BSLZ/p^{k+1} \rightarrow K(K_3(Z/p^k), 3)$. Thus the bulk of the proof of the theorem consists of computing the low dimensional group cohomology of $SLZ/p^k$. Stability results of Wagoner [9] and others mean that it suffices to work with $SL_nZ/p^k$ for large $n$ prime to the order of the group of units in $Z/p^k$. In fact, we will assume that $n$ is large and $n \equiv 1 \mod p$. Our method is based on recursive definition of group extensions and detailed comparison of the resulting Lyndon-Serre spectral sequences.

The key set of extensions are those induced by reduction, $E(k) G_n^k = \ker r_k \longrightarrow_{i_k} SL_nZ/p^k \rightarrow_{\pi_k} SL_nZ/p$.

The initial step in the recursive analysis is provided at $k = 2$ by the calculations of Snaith ($p = 2$), Lluis (odd primes) and Evens and Friedlander ($p > 3$) of the $E_2^{2*}$ terms in the associated spectral sequence with coefficients in $Z$ or $Z/p$. ($G_n^2$ is isomorphic to $M_n^Z Z/p$, the zero trace $n \times n$ matrices over $Z/p$.) A specific resolution-level differential formula is derived, then applied to the spectral sequence $H^*(SL_nZ/2; H^*(M^Z_nZ/2; Z/4)) \Rightarrow H^*(SL_nZ/4; Z/4)$ to complete the determination of $H^4(SL_nZ/4; Z/4)$ and thence of $K_3(Z/4)$. For the odd primes, in particular $p = 3$, spectral sequence pairings and the Charlap and Vasquez [2] differential formula are exploited in order to avoid resolution level calculations in the integral spectral sequences associated with $E(2)$. So for all primes $p$, $H^4(SL_nZ/p^2; Z)$ is known.

For the recursive step, the modules $H^4(G_n^k; Z)$ for $i \leq 4$ and $k > 2$ need first to be estimated. This is done through the spectral sequences associated with the central group extensions

$E(k)$

\[ \begin{array}{c}
M_n^Z Z/p \longrightarrow G_n^k \xrightarrow{\pi_k} G_n^{k-1}, \\
\end{array} \]

Initially take $Z/p$ coefficients. Because the base group in $E(3)$ is an elementary abelian $p$-group, it is straightforward to apply the Hochschild-Serre formula for the $d_2$ differential. In an identical calculation to that which would be used to determine $H^*(M_n^Z Z/p^2; Z/p)$ from the equivalent filtration, the full graded module $H^*(G^3; Z/p)$ is obtained when $p$ is odd. When $p = 2$, an ad hoc computation of desired $E^{**}$ terms must be employed. For $k > 3$ the differential formula cannot be neatly expressed. However, the image of the $d^{0,1}_2$ differential can be shown to be precisely the cokernel of $\pi_{k-1}^*: H^2(G_n^{k-1}; Z/p) \rightarrow H^2(G_n^{k-1}; Z/p)$ by comparing low dimensional terms in the spectral sequence.
associated with \( \hat{E}(k) \) with those in the sequence \( H^*(G_n^{k-2}; H^*(M^*Z/p^2; Z/p)) \Rightarrow H^*(G_n^k; Z/p) \). From this, an isomorphism with the \( k = 3 \) spectral sequence is obtained if \( p \) is odd. For \( p = 2 \), each of the \( H^*(G_n^k; Z/p) \) is isomorphic as \( SL_nZ/p \)-modules if \( k > 3 \), and as groups if \( k \geq 3 \).

The modules \( H^i(G_n^k; Z), i < 4 \), are determined from the \( Z/p \)-results using the integral spectral sequence associated with \( \hat{E}(k) \) and the fact that in this situation, \( d_3(1 \otimes \beta) = 0 \), \( \beta \) the Bockstein \( H^i(-; Z/p) \Rightarrow H^{i+1}(-; Z) \). These modules are expressed in terms of direct summands and quotients of \( H^*(M^*Z/p; Z/p) \), in particular, summands which are the \( (\pi_3 \cdots \pi_k)^* \)-images of \( H^*(G_n^k; Z) \). It is then easy to show that the groups \( H^i(SL_nZ/p; \hat{H}^i(G_n^k; Z)) \) are isomorphic under \( (\pi_3 \cdots \pi_k)^* \) for each \( k \geq 2 \) in total degree less than 6, if \((i, j) \notin \{(0, 5), (1, 4), (2, 3), (0, 4)\} \). Naturality of spectral sequences therefore provides for an isomorphism: \( \ker i_k^* \rightarrow \ker i_k^* \) restricting from the map: \( H^4(SL_nZ/p^2; Z) \rightarrow H^4(SL_nZ/p^k; Z) \) induced by reduction.

To find \( \operatorname{im} i_k^* \), first reconsider the spectral sequences associated with \( \hat{E}(k) \). The \( SL_nZ/p \)-invariants in the \( E_\infty^* \) terms of total degree 4 are determined by specifically examining the action of the differential on invariants in the \( E_2^* \) terms. The Wagoner-Milgram \[10\] result that \( K_3(Z/p), \) defined as \( \lim_k \Pi_3(BGLZ/p^{k+1}) \), contains a copy of the \( p \)-adic integers is interpreted to mean that the subgroup of invariants in \( H^4(G_n^k; Z) \) becomes arbitrarily large with increasing \( k \). By studying possible representatives in \( p \cdot H^4(G_n^k; Z) \) for decreasing \( k \) it can be recursively shown that all \( E_\infty^* \) invariants represent invariants in the full group. With the \( Z/p \) results, we find the invariants in \( H^4(G_n^k; Z) \) to be \( Z/p \oplus Z/p^2 \oplus Z/p^2(k-2) \) if \( p \) is odd, or \( Z/2 \oplus Z/2 \oplus Z/2^2(k-2) \) if \( p = 2 \). Further, \( \pi_k^* \) may be taken to be the zero map on the first summand, and multiplication by \( p^2 \) on the last (and to be an isomorphism between the second summands if \( p = 2 \)).

Next, an injection: \( H^4(SL_nZ/p^k; Z) \rightarrow H^4(SL_nZ/p^{k+1}; Z) \) induced by reduction is established by recursively determining the \( E_2^* \) terms in the spectral sequences \( H^*(SL_nZ/p^k; H^*(M^*Z/p; Z)) \Rightarrow H^*(SL_nZ/p^{k+1}; Z) \), then inspecting differentials. This together with the known action of \( \pi_k^* \) permits the determination of the image of \( i_k^* \), \( k \geq 2 \); it is as shown in the following commutative exact diagram which has now been set up for odd primes \( p \). (The case \( p = 2 \) is entirely analogous.)

\[
\begin{array}{ccc}
\ker i_k^* & \rightarrow & H^4(SL_nZ/p^k; Z) \\
\cong & \uparrow & \cong \\
\ker i_2^* & \rightarrow & H^4(SL_nZ/p^2; Z) \rightarrow \operatorname{im} i_2^* = Z/p.
\end{array}
\]

Then information about \( H^4(SL_nZ/p^2; Z) \) suffices to determine fully \( H^4(SL_nZ/p^k; Z) \) for all \( k > 2 \).
Finally, that $H^5(\text{SL}_n \mathbb{Z}/p^k; \mathbb{Z}) = 0$ when $p > 3$ follows from the natural isomorphisms of the universal coefficient sequences

$$H^4(\text{SL}_n \mathbb{Z}/p^k; \mathbb{Z}) \otimes \mathbb{Z}/p \rightarrow H^4(\text{SL}_n \mathbb{Z}/p^k; \mathbb{Z}/p) \rightarrow \text{Tor}(H^5(\text{SL}_n \mathbb{Z}/p^k; \mathbb{Z}), \mathbb{Z}/p)$$

with the corresponding sequences when $k = 2$. The groups

$$H^i(\text{SL}_n \mathbb{Z}/3; H^j(M_n \mathbb{Z}/3; \mathbb{Z}/3))$$

for $(i, j) = (0, 4)$ and $(2, 2)$ which are needed to obtain $H^4(\text{SL}_n \mathbb{Z}/9, \mathbb{Z}/3)$ are not yet available.

REFERENCES