ORTHOGONAL TRANSFORMATIONS FOR WHICH
TOPOLOGICAL EQUIVALENCE IMPLIES
LINEAR EQUIVALENCE

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Let $R_1, R_2 \in O(n)$, the group of orthogonal transformations of $\mathbb{R}^n$. We say $R_1$ and $R_2$ are topologically (resp. linearly) equivalent if there is a homeomorphism (resp. linear automorphism) $f: \mathbb{R}^n \to \mathbb{R}^n$ such that

\begin{align}
(1) \quad f^{-1}R_1f = R_2: \mathbb{R}^n \to \mathbb{R}^n, \quad f(0) = 0.
\end{align}

(Of course, linear equivalence of $R_1$ with $R_2$ is the same as equality of the respective sets of complex eigenvalues.) The order of an orthogonal transformation is its order as an element of $O(n)$. The purpose of this note is to announce and discuss the proof of the following result [HP].

**Theorem A.** Let $R_1, R_2 \in O(n)$ have order $k = l2^m$, where $l$ is odd and $m > 0$. Suppose that

(a) $R_1$ and $R_2$ are topologically equivalent, and

(b) each eigenvalue of $R_1$ and $R_2$ is either 1 or a primitive $2^m$th root of unity. Then $R_1$ and $R_2$ are linearly equivalent.

If $G$ is a group and $\rho_1, \rho_2: G \to O(n)$ are orthogonal representations, we say $\rho_1$ and $\rho_2$ are topologically (resp. linearly) equivalent if there is a homeomorphism (resp. linear automorphism) $f: \mathbb{R}^n \to \mathbb{R}^n, f(0) = 0$, such that

\begin{align}
(2) \quad f\rho_1(g)(x) = \rho_2(g)f(x),
\end{align}

for all $x \in \mathbb{R}^n, g \in G$. Here is an equivalent statement of Theorem A giving a more geometric description of its condition (b).

**Theorem B.** Let $\rho_1, \rho_2: G \to O(n)$ be orthogonal representations of the finite group $G$ such that $\rho_1|H$ and $\rho_2|H$ define semi-free actions of $H$ on $\mathbb{R}^n$ for each cyclic 2-subgroup $H$ of $G$. If $\rho_1$ and $\rho_2$ are topologically equivalent, then they are linearly equivalent.

Returning to Theorem A, note that if $k$ is odd, condition (b) may be omitted; in this case the result has been proved independently, using rather different methods, by Madsen and Rothenberg [MR]. If $k$ is an odd prime power,
Theorem A had been proved in [Sc] and, if \( k \leq 6 \), in [KR] where the more general question of linear versus topological equivalence of arbitrary linear endomorphisms of \( \mathbb{R}^n \) was studied. (In fact, the general question was reduced to the special case of orthogonal transformations of finite order in [KR].)

Unless \( k = 4 \), our result is the best possible, in the following sense. The remarkable results of [CS1] include for each \( k = \ell 2^m \), where \( m \geq 2 \), examples of topologically equivalent orthogonal transformations \( R_1 \) and \( R_2 \) of order \( k \), where \( R_1^j \) and \( R_2^j \) each have eigenvalues of any prescribed order \( 2^j \), \( 0 \leq j \leq m \), with at least one where \( 1 < j < m \), and where \( R_1 \) and \( R_2 \) are not linearly equivalent.

Roughly stated, the proof of Theorem A has two parts. First, the theory of Anderson and Hsiang [AH1–3], which describes the obstructions to making \( f \) piecewise-linear (p.l.), is applied with the consequence that in (1) above, \( R_1 \) and \( R_2 \) may be assumed to have no eigenvalues equal to 1 and \( f \) may be assumed p.l. on \( \mathbb{R}^n - \{0\} \). Now add a point at infinity to the range of \( f \), discard the origin (getting \( \mathbb{R}^n \) again), and take the union of this with the domain of \( f \), identifying corresponding points under \( f|\mathbb{R}^n - \{0\} \). The result is p.l. homeomorphic to the \( n \)-sphere \( S^n \), and \( R_1 \) and \( R_2 \) conspire to define a periodic p.l. map \( R: S^n \to S^n \) of period \( k \).

If \( G \) denotes the cyclic group of order \( k \), then we have constructed a p.l. \( G \)-action \( G \times S^n \to S^n \) with exactly two points \( x_1, x_2 \) fixed by \( G \), near which \( G \) is actually acting smoothly, with its generator \( R \) inducing \( R_i \) on the tangent space to \( x_i \). Now if \( G \) were acting everywhere smoothly on \( S^n \), then the Atiyah-Singer \( G \)-signature formula (ASGSF) might be used to show the eigenvalues of \( R_1 \) and \( R_2 \) were the same: this general line of argument seems first to have been used by Atiyah, Bott and Milnor [AB, 7.15, 7.27].

On the other hand, the topological equivalence of linearly inequivalent \( R_i \in O(9) \) constructed in [CS2] was likewise p.l. on \( \mathbb{R}^9 - \{0\} \). The essential difference is that in this case the ASGSF gives no information about the eigenvalues at isolated fixed points because of the presence of \((-1)\)-eigenvalues (the Euler class of the \((-1)\)-eigenbundle must vanish in [AS, 6.12]). Moreover, condition (b) in Theorem A avoids \((-1)\)-eigenvalues on all powers of the \( R_i \), unless that power has order \( 2\ell \), \( \ell \) odd.

Thus, in the presence of a p.l. ASGSF, we can make the argument of [AB, 7.27] work to give Theorem A. To get such a result, we first define a bordism theory of p.l. \( G \)-actions on closed p.l. manifolds, requiring (in place of the slice and tube theorems in the differentiable case) that \( G \) preserve p.l. block bundles around orbit types. The resultant bordism groups are "computable" in a way similar to that exposed in [CF] (again the smooth analogue). Given such a p.l. \( G \)-action \( G \times M \to M \), one defines the \( G \)-signature representation as in [AS, §6].
We then show by bordism computations that the trace of this representation on a generator $g$ of $G$ depends only on the fixed point set of $g$ and its equivariant normal (block) bundle, and that this dependency is sufficient to detect the eigenvalues at the fixed points for the p.l. $G$-action on $S^n$ constructed above.

REFERENCES


[HP] W.-C. Hsiang and W. Pardon, When are topologically equivalent orthogonal representations linearly equivalent? (preprint).


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