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Bases in Banach spaces. II, by Ivan Singer, Springer-Verlag, Berlin and New York, 1981, viii + 880 pp., $78.00.

Introduction. In his book *Bases in Banach spaces*. II (BBS II), Ivan Singer takes all knowledge of bases and their generalizations to be his province. More precisely, he states in the preface that "this volume attempts to present the results known today on generalizations of bases in Banach spaces and some unsolved problems concerning them". *Bases in Banach spaces*. I (BBS I) was published in 1970 and BBS II in 1981. During the writing of these books, basis theory and its generalizations began to develop very rapidly. The task of the author became not that of describing a theory already essentially developed, but of presenting a theory in a very rapid state of development. Thus, in order to achieve his goal of a complete account of basis theory, its generalizations, and its applications, Ivan Singer is working on a third volume on applications, bases in concrete spaces, and perhaps some loose ends.

The book under review, BBS II, is encyclopaedic (at the Banach space level) with respect to its subject. It consists of twenty-one sections plus a section entitled *Notes and remarks*. The review will discuss the section on the solution

of the basis problem and the section on basic sequences separately. The remaining nineteen sections of the book are on the known generalizations of bases. From these nineteen sections the reviewer will extract only one of several possible themes for discussion, namely: What are some of the main generalizations of bases that have arisen over the years, how general are they, and how are they related to one another?

The basis problem. Recall that a (*Schauder*) *basis* of an infinite dimensional Banach space (*B*-space) E is a sequence $\{x_n\}$ in E such that, for each x in E, there exists a unique sequence of scalars $\{a_i\}$ such that $x = \sum_{i=1}^{\infty} a_i x_i$, convergence in the norm topology. For almost forty years the question of Banach ([**2**], 1932), "Does every separable *B*-space possess a basis?" stood open. A *B*-space E is said to have the *approximation property* if the identity operator on E can be approximated uniformly on every compact subset of E by continuous linear operators of finite rank. If each of the above continuous linear operators has norm no greater than some positive number λ, the space E is said to have the *bounded approximation property*. When P. Enflo ([**10**], 1973) showed the existence of a separable reflexive *B*-space without the approximation property he also settled the basis problem in the negative. Enflo's result and his methods inspired a flow of papers on separable *B*-spaces which fail to have the approximation property. BBS II begins with a section zero which gives two different proofs of the Pełczyński-Figiel ([**29**], 1973) theorem on the existence of subspaces of c_0 and l_p $(2 < p < \infty)$ which fail to have the approximation property. These proofs are refinements of the proofs of Figiel [**11**] and Davie ([**5**], 1973), respectively. Section zero ends with the Lindenstrauss-James ([**24**], 1971) example of a *B*-space E with a basis whose dual space E^* is separable but does not have the approximation property. This example is used later to construct the decisive example of Figiel and Johnson ([**12**], 1973) of a separable *B*-space with the approximation property but without the bounded approximation property. Thus there are separable *B*-spaces with the approximation property which do not possess a basis. Singer's detailed and clear exposition of the proofs of the results mentioned above together with his copious notes and remarks on these outstanding achievements are a valuable contribution to the literature.

Basic sequences. The negative solution of the basis problem not only stimulated research on bases but also provided motivation for studying their generalizations. The first of these is that of basic sequence. A sequence $\{x_n\}$ in a Banach space is a *basic* sequence if it is a basis of its closed linear span $[x_n]$. Banach ([**2**], 1932) observed without proof that such sequences exist in every *B*-space. Recall that a basis is *unconditional* if the series's convergence in the expansion of each element is unconditional. Bessaga and Pełczyński ([**4**], 1958) asked the question, "Do unconditional basic sequences exist in every *B*-space?" This remains as probably the most important of the open questions in basis theory. A related question which goes back to Banach's book is: "Does every infinite dimensional *B*-space have a subspace isomorphic to either c_0 or l_p, for some $1 \leqslant p < \infty$?" Since c_0 and the l_p spaces have unconditional bases, a positive solution to this question would have provided a positive solution to

the existence of unconditional basic sequences. However, Tsirelson ([**37**], 1974) gives an example of a reflexive B-space which contains no subspace isomorphic to c_0 or l_p, $1 \leqslant p < \infty$. In contrast to our ignorance about the existence of unconditional basic sequences, it was shown by Pełczyński and Singer ([**30**], 1964) that every B-space with a basis has a conditional basis and from this it follows that every B-space contains a conditional basic sequence.

Bessaga and Pełczyński ([**3**], 1958) wrote a pioneering paper in which sufficient conditions are given for a sequence in a B-space E to contain a basic subsequence. Johnson and Rosenthal ([**19**], 1972) discovered a dual to the main result of Bessaga and Pełczyński. It gives sufficient conditions for a sequence in E^* to contain a subsequence which is a w^*-Schauder basic sequence. Yet another interesting existence theorem is due to Davis, Dean, and Lin ([**6**], 1973) who show that *for every B-space E there exists a biorthogonal system* $\{x_n, f_n\}$, $(\{x_n\} \subset E, \{f_n\} \subset E^*)$ *such that* $\{x_n\}$ *is basic in* E, $\{f_n\}$ *is basic in* E^* *and each sequence is norm bounded*. Perhaps the most interesting result in this area is due to Kadec and Pełczyński ([**21**], 1965) who showed that *a sequence* $\{x_n\}$ *in a B-space contains a basic subsequence if and only if one of the following holds*: (i) $\{x_n\}$ *is not conditionally weakly compact*; (ii) 0 *is a weak limit point of* $\{x_n\}$.

Biorthogonal systems. Recall that if E is a B-space a *biorthogonal system* is a pair of sequences $(\{x_n\}, \{f_n\})$, $\{x_n\} \subset E$, $\{f_n\} \subset E^*$, such that $f_i(x_j) = \delta_{ij}$, $i, j = 1, 2, \ldots$. A sequence $\{x_n\}$ is part of a biorthogonal system if and only if $x_n \notin [x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots]$, $n = 1, 2, \ldots$. Such a sequence is called a *minimal* sequence. Minimal sequences exist in every B-space. For a minimal sequence $\{x_n\}$ there is a unique sequence of functionals (the coefficient functionals) $\{f_n\}$ which form a biorthogonal pair with $\{x_n\}$ if and only if $\{x_n\}$ is complete. In each separable B-space E, every minimal sequence has an extension to a minimal sequence which is complete in E.

Arsove and Edwards ([**1**], 1960) called a biorthogonal system $\{x_n, f_n\}$, $(\{x_n\} \subset E, \{f_n\} \subset E^*)$ with $\{f_n\}$ total, a *generalized basis* of E. A space need not be separable to have a generalized basis, e.g. l_∞ has one. In fact a B-space E has a generalized basis if and only if E^* contains a total sequence $\{f_n\}$.

A biorthogonal system $\{x_n, f_n\}$, $(\{x_n\} \subset E, \{f_n\} \subset E^*)$ with $\{x_n\}$ complete and $\{f_n\}$ total is called an M-basis (Markuševič-basis). Banach ([**2**], 1932) observed, without proof, and Markuševič ([**27**], 1943) proved that every separable B-space has an M-basis. Banach ([**2**], 1943) asked if, for every separable B-space E, there exists an M-basis $\{x_n, f_n\}$ of E such that both $\{x_n\}$ and $\{f_n\}$ are norm bounded? This was finally answered in the affirmative by Ovsepian and Pełczyński ([**28**], 1975). With the help of others their result has been sharpened to the following: *In every separable B-space E and for all* $\varepsilon > 0$ *there exists an M-basis* $\{x_n\}$ *with coefficient functionals* $\{f_n\}$ *such that* $\sup \|x_n\| < 1 + \varepsilon$ *and* $\sup \|f_n\| < 1 + \varepsilon$. It is interesting to note that Davis and Johnson ([**7**], 1973) have shown that if "M-basis" is replaced above by "complete minimal sequence" the conclusion is "$\|x_n\| \leqslant 1 + \varepsilon$ and $\|f_n\| = 1$ for each n". On the other hand, Singer ([**36**], 1971) and Davis and Johnson ([**7**], 1973) have shown that if "M-basis" is replaced by "biorthogonal system $\{x_n, f_n\}$ with $\{f_n\}$ total (i.e. $\{x_n, f_n\}$ is a generalized basis)" then the conclusion is "$\|x_n\| = 1$ and $\|f_n\| \leqslant 1 + \varepsilon$ for each n".