BROWNIAN MOTION, GEOMETRY, AND GENERALIZATIONS OF PICARD’S LITTLE THEOREM

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ABSTRACT. Brownian motion is introduced as a tool in Riemannian geometry, and it is shown to be useful in the function theory of manifolds, as well as in the study of maps between manifolds. As applications, a generalization of Picard’s little theorem, and a version of it for Riemann surfaces of large genus are given.

1. Picard’s theorem for nonhyperbolic manifolds. Let $M$ and $N$ be complete Riemannian manifolds with metrics $g_M$, $g_N$, resp. Assume $F: M \rightarrow N$ is a $C^2$ map. $F$ is said to be harmonic \cite{2} if its second fundamental form has trace 0. Define the tensor

$$\xi^{\alpha\beta}(x) = g_{ij} \left[ \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \right](x), \ x \in M.$$ 

Since $(\xi^{\alpha\beta}(x))$ is a symmetric matrix, its eigenvalues are nonnegative, and we may order them as follows: $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x) \geq 0$. $F$ is said to be $K$-quasiconformal \cite{5} if $\lambda_1(x) \leq K^2 \lambda_n(x)$ for all $x \in M$.

We define polar coordinates $(r, \theta)$ on $N$ via the exponential map. There will be two restrictions on the curvature of $N$:

(i) The sectional curvatures of $N$ are bounded below by $-L^2 < 0$.

(ii) Each of the sectional curvatures at $(r, \theta) \in N$ determined by $dr$ and some other tangent vector, is bounded above by $K(r)$, where $K(r)$ satisfies (a) for some $\varepsilon > 0$, $-K(r) \sim r^2 e^{-2}$; (b) there exists a $C^\infty$ solution $u(r)$ of the equation

$$u''(r) = K(r)u(r), \quad u(0) = 0, \quad u'(0) = 1,$$

and $u'(r)$ is always positive.

(Note that such a solution can always be found if $K(r)$ is negative.)

THEOREM 1. Suppose $M$ and $N$ are as above with the curvature of $N$ satisfying (i) and (ii). Then, if Brownian motion on $M$ has trivial tail $\sigma$-field, every $K$-quasiconformal harmonic map $F: M \rightarrow N$ is constant.

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2. Proof of Theorem 1. The following lemmas are essential.

**Lemma 2.** Let $X_t$ be Brownian motion on $M$, with $X_0$ chosen so that $F(X_0) = x_0 \in N(x_0)$. Then, there is a random time change $\alpha(t) = \int_0^t ds / \lambda_1(x_s)$ and a constant $C > 0$, such that if $\rho_t = r \circ F(X_{\alpha(t)})$, then for $t > \tau$ and $\rho_t(\tau) \neq 0$, $d\rho_t(\tau)/dt = a(\tau)(X_{\alpha(t)})dB_t + b(\tau)(X_{\alpha(t)})dt$, where $1/K \leq |a(\tau)(X_{\alpha(t)})| \leq 1$ and $b(\tau)(X_{\alpha(t)}) \geq C d_t^{1/2} - 1$ if $\rho_t$ is larger than some constant $R$.

The statement uses the stochastic calculus discussed in [8]. Let $\tau$ be a stopping time for $X_{\alpha(t)}$.

**Lemma 3.** If $\rho_t^{(\tau)}$ is the distance of $F(X_{\alpha(t)})$ from $F(X_{\alpha(\tau)})$, then for $t > \tau$ and $\rho_t(\tau) \neq 0$, $d\rho_t^{(\tau)} = a^{(\tau)}(X_{\alpha(t)})dB_t^{(\tau)} + b^{(\tau)}(X_{\alpha(t)})dt$, where $1/K \leq |a^{(\tau)}(X_{\alpha(t)})| \leq 1$ and $|b^{(\tau)}(X_{\alpha(t)})| \leq (\eta/2)L_0 \coth L_0^{1/2} + 1$.

The proofs of Lemmas 2 and 3 require Ito’s lemma, the $K$-quasiconformal condition, and the Hessian comparison theorem of Greene and Wu [3]. Lemma 2 is used to determine the speed with which $\rho_t$ goes to $\infty$.

**Lemma 4.** \( \liminf_{t \to \infty} \rho_t/t^{2/4 - \epsilon} > c > 0 \).

Lemma 4 will be required in the proof of

**Lemma 5.** Let $\tau_Q$ be the first time that $\rho_t > Q$. Then, for $h$ sufficiently large, there is a $q \in (0, 1)$ and a constant $C_1$ such that

\[
P\{\rho_t + \tau_Q > C_1 (t + h)^{2/4 - \epsilon} \mid X_{\alpha(\tau_Q)}\} > q\]

uniformly in $X_{\alpha(\tau_Q)}$.

**Lemma 6.** Let $r = \inf(r(x), r(y))$ for $x, y \in N$. If $|\Theta(x, y)|$ denotes the angular distance between $\theta(x)$ and $\theta(y)$, then for some constants $C$ and $R$, $r > R$,

\[
|\Theta(x, y)| \leq C \frac{N_d(x, y)}{\exp(r^2)}
\]

where $N_d$ denotes the distance in the manifold $N$.

**Proof.** We use the Rauch comparison theorem for the same comparison manifold $P$ as is used in the proof of Lemma 2.

**Lemma 7.** There exists a random time $T$ such that for all $m > T$,

\[
\sup_{\alpha(m) \leq s \leq \alpha(m + 1)} N_d(F(X_s), F(X_{\alpha(m)})) \leq m.
\]

**Proof.** Using Lemma 3, we must show, for $m > T$, that $\sup_{t \leq s \leq m + 1} \rho_t^{(m)} \leq m$. The argument is similar to an idea of Prat [11], and is also used by Kendall [7] and Pinsky [10].
**Lemma 8.** Fix $\delta > 0$. If $\tau$ is a stopping time, we may choose an integer $J$ so large that

$$P\{\sup_{\tau - m \leq t \leq \tau + 1} \rho_t^{(\tau + m)} \leq m + J, \text{ for all } m \geq 1\} \geq 1 - \delta.$$ 

Lemmas 4–8, and a result of Stroock and Varadhan [12, Theorem 3.1] show that the tail $\sigma$-field $\lim_{\tau \to \infty} \Theta(F(X_\tau))$ of $X_\tau$ exists and is nontrivial. This is a contradiction, so $F$ must be constant.

3. **Picard's theorem for Riemann surfaces of large genus.** In the sequel, let $M$ and $N$ be compact Riemann surfaces with a finite number of points deleted. Any such surface $N$ is homeomorphic to a sphere with $t(N)$ tori attached and $p(N)$ points deleted. Let $n(N) = 2t(N) + p(N)$.

**Theorem 2.** Let $F: M \to N$ be a holomorphic map. If $t(M) > 0$, $p(M) > 0$, $p(N) > 1$, and $n(N) - n(M) > 0$, then $F$ is constant.

**Proof of Theorem 2.** The proof uses Brownian motion on Riemann surfaces, which can be defined as follows. Let $\{m_i\}$ be coordinate patches on $M$, and for each $i$, let $c_i: m_i \to C$, where $C$ is the complex plane, be a holomorphic map. For $p \in M$, choose a patch $m_{i(1)}$ containing $p$. Let $W(t)$ be Brownian motion in $C$, and let $\sigma_1$ be the first time that $p + W(t)$ hits the boundary of $c_{i(1)}(m_{i(1)})$. For $0 < t < \sigma_1$, let $B(t) = c_{i(1)}^{-1}(p + W(t))$. Next, let $m_{i(2)}$ be a patch containing $B(\sigma_1)$, and let $\sigma_2$ be the first time that $c_{i(2)}(B(\sigma_1)) + W(t) - W(\sigma_1)$ hits the boundary of $m_{i(2)}$. For $0 < \sigma_1 < t < \sigma_2$, let

$$B(t) = c_{i(2)}^{-1}(c_{i(2)}(B(\sigma_1)) + W(t) - W(\sigma_1)),$$

and note that we have defined $B(t)$ to be continuous at $\sigma_1$. For $k > 2$, define $\sigma_k$ and $B(t)$, $\sigma_k \leq t \leq \sigma_{k+1}$, analogously. We also need

**Lemma 9.** Let $p(N) > 0$. Then the fundamental group of $N$ is the free group with $n(N) - 1$ generators. The generators may be taken to be the cycles around single points (except one) and the canonical generators of the tori.

Suppose $F$ is not constant. Let $G(M)$ be the fundamental group of $M$ with base point $p$, and let $G(N)$ be the fundamental group of $N$ with base point $F(p)$. Let $\{\alpha_i^M\}$, $\{\alpha_i^N\}$ be the generators of $G(M)$, $G(N)$, resp., as described in Lemma 9. Choose 2 generators $\alpha_1^M$, $\alpha_2^M$ of $G(M)$ which are generators of a torus, and let $\alpha_3^M$, $\ldots$, $\alpha_{n(M)}^M$ be the remaining generators. Let $H(M)$ be the smallest normal subgroup of $G(M)$ containing $\alpha_3^M$, $\ldots$, $\alpha_{n(M)}^M$ and the commutator $[G(M), G(M)]$. Let $\hat{G}(M) = G(M)/H(M)$, and note that $\hat{G}(M)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. 

Let $\mathcal{O}$, $\mathcal{O}_2$, $\mathcal{O}_3$ be neighborhoods of $p$, contained in $m^{(1)}$, whose images under $c^{(1)}$ are discs of radii $e$, $2e$, $3e$, respectively. Let $B^Q(t)$ be the Brownian motion on $M$ whose initial distribution $B(0)$ is the measure $Q$, concentrated on the boundary of $\mathcal{O}$. Let $\tau_0 = 0$; given $\tau_i$, let $\tau_{i+1}$ be the first time after $\tau_i$ for which $B(\tau_{i+1}) \in \partial \mathcal{O}$, and $\{B(t): \tau_i < t < \tau_{i+1}\}$ corresponds to a nontrivial element of $\hat{G}(M)$. Note that since $M$ is compact except for deleted points, all $\tau_i$ are finite.

Next, let $x_n = B^Q(\tau_n)$, and note that $x_n$ is a Markov process with respect to the fields $\sigma \{B(t): t < \tau_n\}$. Let $m$ be the measure on $\partial \mathcal{O}$ induced by Lebesgue measure on the circle $c^{(1)}(\partial \mathcal{O})$. By a theorem of Harris [6], it can be shown that $x_n$ has an invariant measure. We set $Q$ equal to this measure.

Let $\hat{X}_i$ be the element of $\hat{G}(M)$ corresponding to $\{B^Q(t): 0 < t < \tau_i\}$. Since $\hat{G}(M) \cong \mathbb{Z} \times \mathbb{Z}$, we may regard $\hat{X}_i$ as a random walk on $\mathbb{R}^2$. Let $\Delta_i = \hat{X}_i - \hat{X}_{i-1}$. Then, $\{\Delta_i\}$ is a stationary process with $E|\Delta_i|^4 < \infty$. Moreover the process is symmetric, so $\Delta_i$ and $-\Delta_i$ have the same distribution. Using the central limit theorem (see Theorem 9 of Phillip [9]), it is shown that $\hat{X}_i$ is recurrent. By Lemma 3.1 of [1], it follows that $X_i$ is recurrent.

Now, $F$ induces a map $F_H$ from $H/[H, H]$ to $G(N)/[G(N), G(N)]$. Since these are commutative groups with difference in dimension at least 3, it follows that there must be at least 3 generators $\alpha_1^N, \alpha_2^N, \alpha_3^N$ of $G(N)$ which generate a subgroup of $G(N)/[G(N), G(N)]$ isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Let $H(N)$ be the smallest normal subgroup of $G(N)$ containing $\alpha_4^N, \ldots, \alpha_{n(N)-1}^N$ and $[G(N), G(N)]$, and let $\hat{G}(N) = G(N)/H(N)$. Then, $F$ induces a map $\hat{F}$ from $\hat{G}(M)$ to $\hat{G}(N)$.

Using Brownian motion on $N$, we construct as before an invariant measure $Q$ on $F(\mathcal{O})$ and a random walk $\hat{Y}_i$ from $F(B(t))$. By a theorem of Lévy [1], $F(B(t))$ is Brownian motion on $N$ with a new time scale. By the Borel-Cantelli lemma, $\hat{Y}_i$ is transient. Davis’s argument [1] then shows that the random walk $Y_i$ induced by $F(B(t))$ with $B(0) = p$ is also transient. But then $\hat{F}(X_i) = Y_i$, and $X_i$ is recurrent. This contradiction shows that $F$ must be constant.

REFERENCES

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